Polytopes, Toric Varieties, and Ideals

A Look at Pyramids, Prisms, and Products

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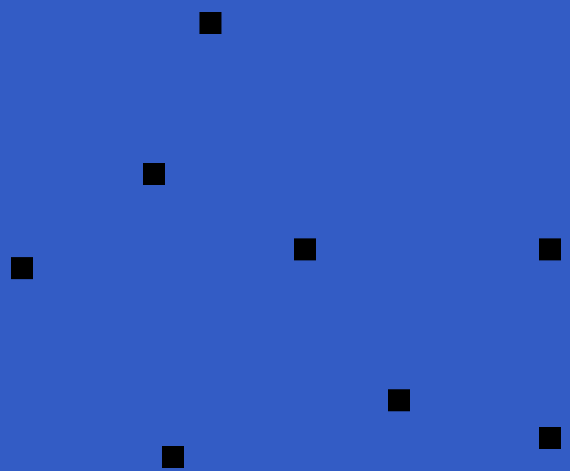
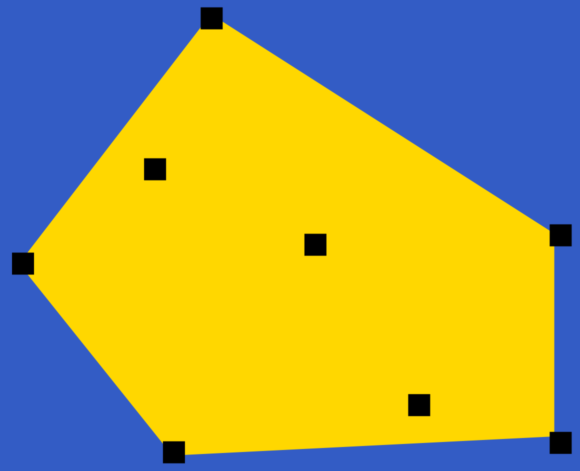
Summer 2005

Polytopes

Take a finite collection of elements in $\mathbb{Z}^n \subset \mathbb{R}^n$. A lattice polytope is the convex hull of these integer points.



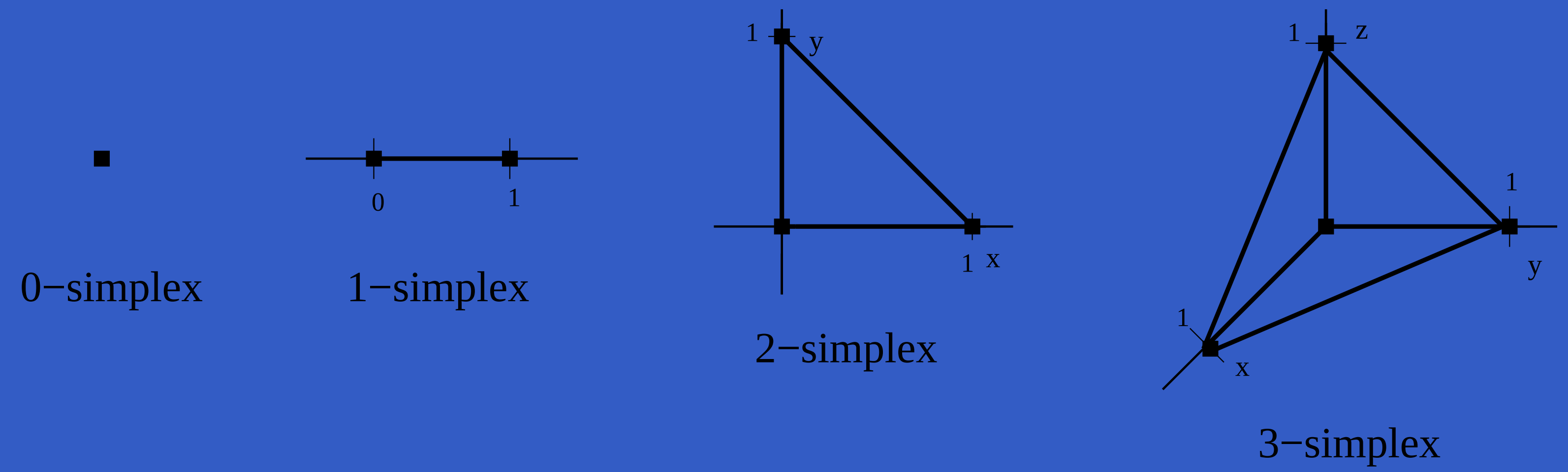
Finite Set of Points



Lattice Polytope

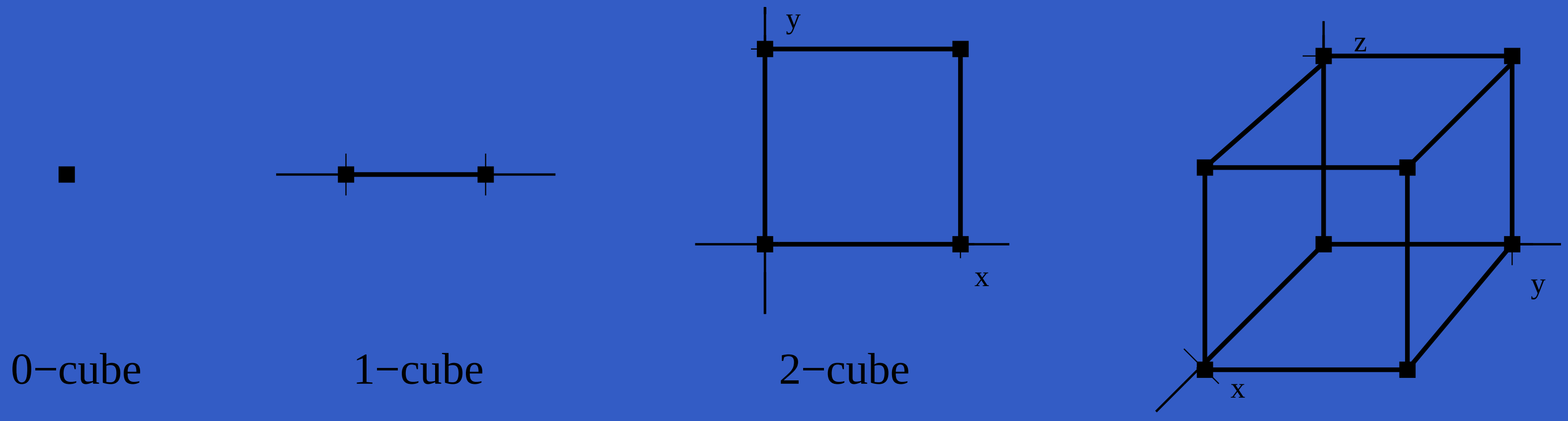
Examples of Polytopes: Simplices

Definition: A standard lattice n-simplex, $\Delta_n \subset \mathbb{R}^n$, is the convex hull of 0 and the standard basis vectors.



Examples of Polytopes: Cubes

Definition: The standard n-cube $\subseteq \mathbb{R}^n$ is the convex hull of all points whose coordinates are made up of 0's and 1's.



0−cube 1−cube 2−cube 3−cube

Defining a Map

Definition: Let K be a field. Let $\underline{t} = (t_1, t_2, ..., t_n) \in (K^*)^n$ and let $\underline{a} = (a_1, a_2, ..., a_n) \in \mathbb{Z}^n$. Define $\underline{t}^{\underline{a}} = t_1^{a_1} t_2^{a_2} ... t_n^{a_n}$.

Example: $(x, y)^{(1,2)} = xy^2$

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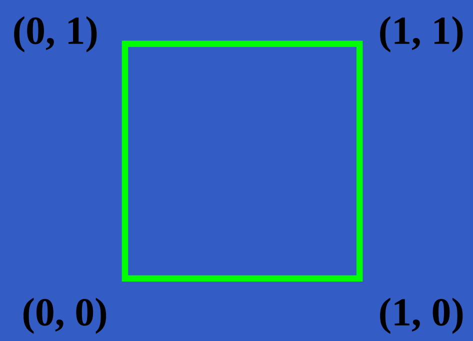
Definition: Let P be a lattice polytope $\subset \mathbb{R}^n$ with $P \cap \mathbb{Z}^n = \{\underline{a}_0, ..., \underline{a}_m\}$. $\phi_P : (\mathbb{C}^*)^n \longrightarrow \mathbb{P}^m$, where $\phi_P(\underline{t}) = [\underline{t}^{\underline{a}_0} : ... : \underline{t}^{\underline{a}_m}]$.

Example 1

Let P = 2-cube.

$P \cap \mathbb{Z}^2 = \{(0,0),(1,0),(0,1),(1,1)\}$.



$\phi_P : (\mathbb{C}^*)^2 \longrightarrow \mathbb{P}^3$, $\phi_P((t_1,t_2)) = [1 : t_1 : t_2 : t_1 t_2]$.

Im$(\phi_P) = \{[\lambda : \lambda t_1 : \lambda t_2 : \lambda t_1 t_2] \mid \lambda \neq 0\}$, as we account for homogeneous coordinates.

Toric Varieties and Ideals

Definition: Let P be a n-dimensional lattice polytope $\subseteq \mathbb{R}^n$ with $| P \cap \mathbb{Z}^n |$ = $m + 1$. The closure of the image of ϕ_P in $\mathbb{P}^m_{\mathbb{C}}$ is the projective toric variety X_P.

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Definition: The ideal of X_P is $I(X_P) = \{f \in \mathbb{C}[x_0, ..., x_m] \mid f(\underline{a}) = 0, \; \forall \underline{a} \in X_P\}$.

Toric Varieties and Ideals

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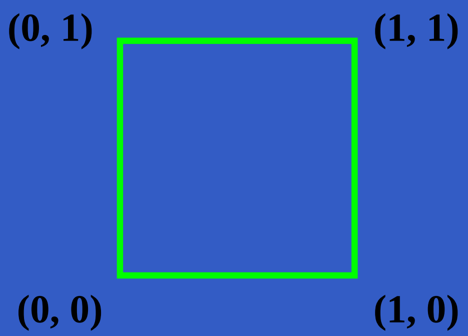
Proposition: Let $f \in \mathbb{C}[x_0, ..., x_m]$.
$f(\underline{a}) = 0,\ \forall \underline{a} \in X_P$ if and only if $f(\underline{b}) = 0,\ \forall \underline{b} \in \mathrm{Im}\phi_P$.

Example 1 continued

Im$(\phi_P) = \{[\lambda : \lambda t_1 : \lambda t_2 : \lambda t_1 t_2] \mid \lambda \neq 0\}$.
$X_P = \overline{Im(\phi_P)}$.

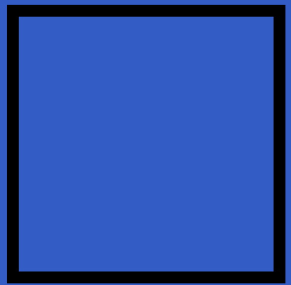


$I(X_P) = \{f \in \mathbb{C}[x_0, .., x_3] \mid f([\lambda : \lambda t_1 : \lambda t_2 : \lambda t_1 t_2]) = 0\}$
where $\lambda \neq 0$.

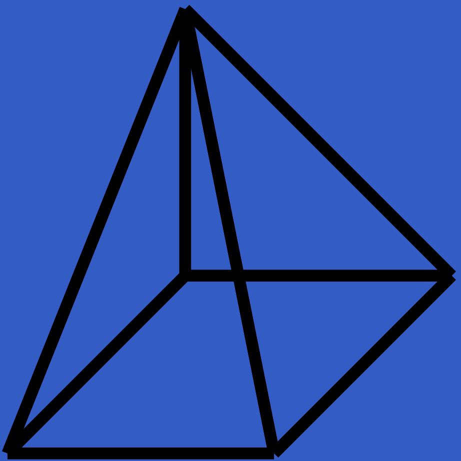
Using Macaulay2, $I(X_P) = \langle x_1 x_2 - x_0 x_3 \rangle$

Pyramids

Definition: Let P be a d-dimensional polytope $\subset \mathbb{R}^n$, $n > d$. The pyramid of P, pyr(P), is convex hull of P and $q \in \mathbb{R}^n$ where $q \notin$ aff(P).



P



pyr(P)

Pyramids and Ideals

Let P $\subseteq \mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^{n+1}$ be a lattice polytope, $\mid P \cap \mathbb{Z}^n \mid = m + 1$.

Let pyr(P)$\subset \mathbb{R}^{n+1}$ equal convex hull of P and $\underline{q}$ where $\underline{q} = (q_1, ..., q_n, 1)$.

Proposition: $I(X_P)$ and $I(X_{pyr(P)})$ have the same generators.

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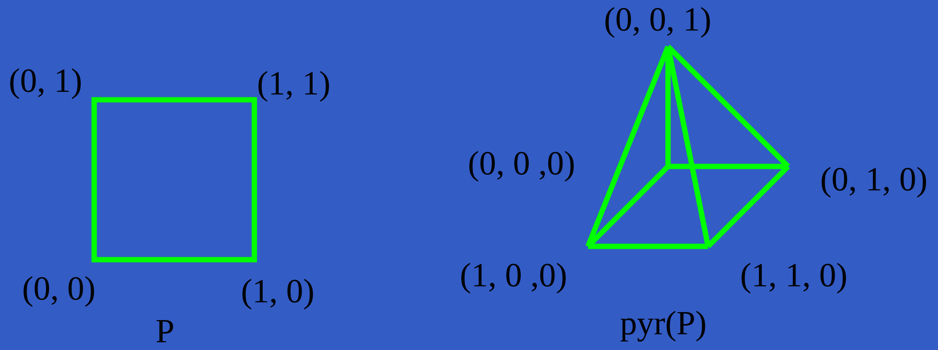
<u>**Proposition**</u>: $I(X_P)$ and $I(X_{pyr(P)})$ have the same generators.

<u>**Remark**</u>: $I(X_P) \subseteq \mathbb{C}[x_0, ..., x_m]$ and $I(X_{pyr(P)}) \subseteq \mathbb{C}[x_0, ..., x_{m+1}]$.

Idea of Proof

Let P = 2-cube. Let $\underline{q} = (0, 0, 1)$. $I(X_P) = \langle x_1x_2 - x_0x_3 \rangle$ and Im($\phi_P) = \{[\lambda : \lambda t_1 : \lambda t_2 : \lambda t_1t_2] \mid \lambda \neq 0\}$.



Im($\phi_{pyr(P)}) = \{[\lambda : \lambda t_1 : \lambda t_2 : \lambda t_1t_2 : \lambda t_3] \mid \lambda \neq 0\}$

What do elements of $I(X_{pyr(P)})$ look like?

Attempts

Let's try to form a polynomial $f \in \mathbb{C}[x_0, ..., x_4]$ such that $f([\lambda : \lambda t_1 : \lambda t_2 : \lambda t_1 t_2 : \lambda t_3]) = 0$.

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Attempt 1: $x_0 x_3 - x_2\, x_4$ = $\lambda^2 t_1 t_2 - \lambda^2 t_2 t_3 \neq 0$

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Attempt 2: $x_1 x_2 x_4 - x_0 x_4^2$ = $\lambda^3 t_1 t_2 t_3 - \lambda^3 t_3^2 \neq 0$

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Attempt 2: $x_1 x_2 x_4 - x_0 x_4^2$ = $\lambda^3 t_1 t_2 t_3$ - $\lambda^3 t_3^2 \neq 0$

Attempt 3:
$x_3 x_0 x_4^2 - x_3 x_2 x_4^2$ = $\lambda^4 t_1 t_2 t_3^2 - \lambda^4 t_1 t_2^2 t_3^2 \neq 0$

Attempts

Let's try to form a polynomial $f \in \mathbb{C}[x_0, ..., x_4]$ such that $f([\lambda : \lambda t_1 : \lambda t_2 : \lambda t_1 t_2 : \lambda t_3]) = 0$.

Attempt 3:
$x_3 x_0 x_4^2 - x_3 x_2 x_4^2 = \lambda^4 t_1 t_2 t_3^2 - \lambda^4 t_1 t_2^2 t_3^2 \neq 0$

Why didn't Attempt 3 work?

$$x_3 x_0 x_4^2 - x_3 x_2 x_4^2 = x_4^2 (x_3 x_0 - x_3 x_2)$$
$$= \lambda^2 t_3^2 (\lambda^2 t_1 t_2 - \lambda^2 t_1 t_2^2)$$

Problem: $\lambda^2 t_1 t_2 - \lambda^2 t_1 t_2^2 \neq 0$.

In other words: $x_3 x_0 - x_3 x_2 \notin I(X_P)$

Attempts

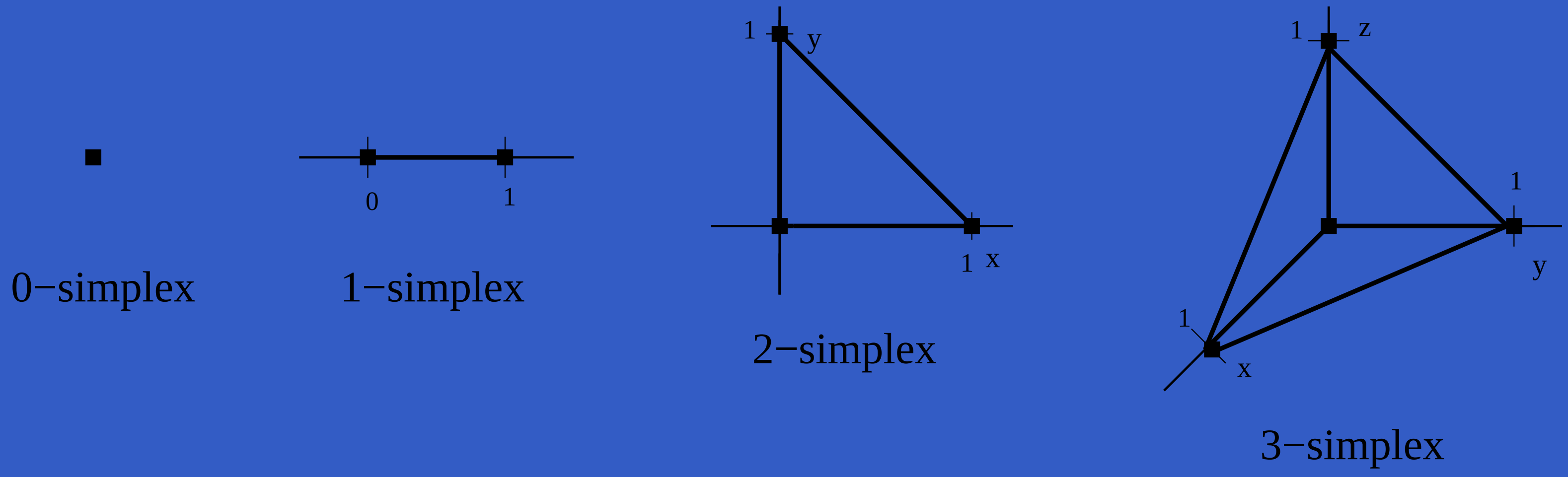
Let's try to form a polynomial $f \in \mathbb{C}[x_0, ..., x_4]$ such that $f([\lambda : \lambda t_1 : \lambda t_2 : \lambda t_1 t_2 : \lambda t_3]) = 0$.

Recall $I(X_P) = \langle x_1 x_2 - x_0 x_3 \rangle$

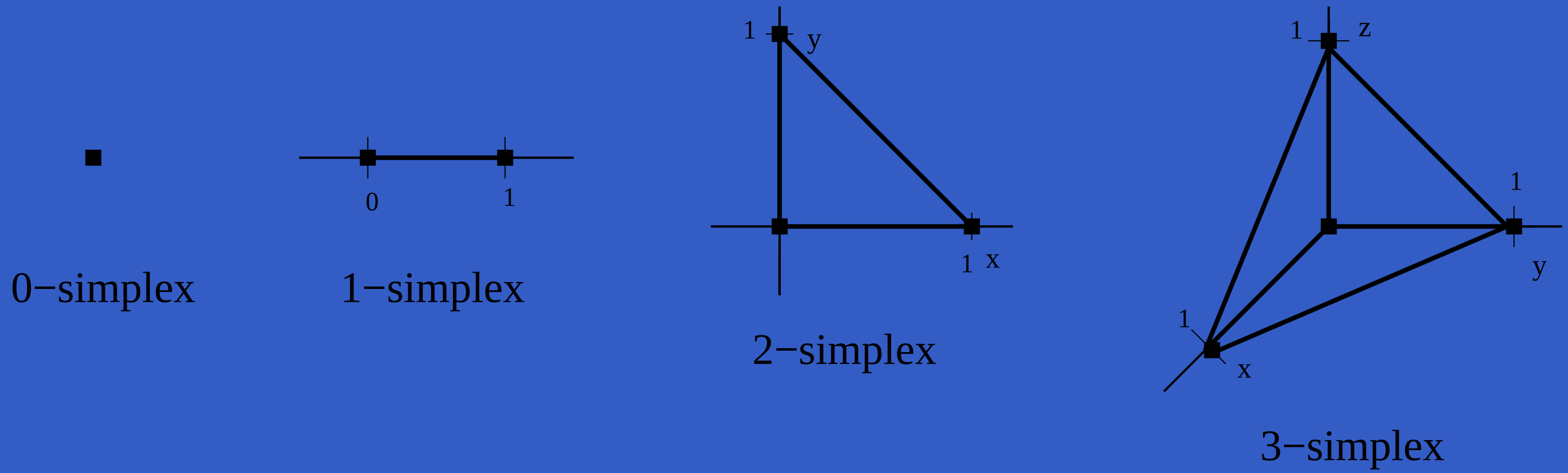
Attempt 4:
$x_4^2(x_1 x_2 - x_0 x_3) = \lambda^2 t_3^2 (\lambda^2 t_1 t_2 - \lambda^2 t_1 t_2) = 0$

Thus $f = x_4^k\, h(x_0, ..., x_3)$ where $k \in \mathbb{N}$ and $h \in I(X_P)$.

Back to Simplices



Back to Simplices

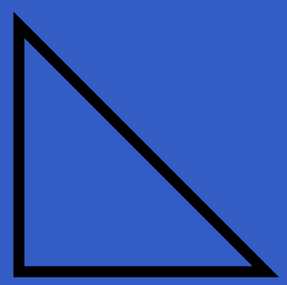


Thus $I(X_{\Delta_0}) = I(X_{\Delta_1}) = \ldots = I(X_{\Delta_n}) = \ldots$.
Im$\phi_{\Delta_0} = \{[\lambda] \mid \lambda \neq 0\}$.
$I(X_{\Delta_0}) = \langle 0 \rangle$.

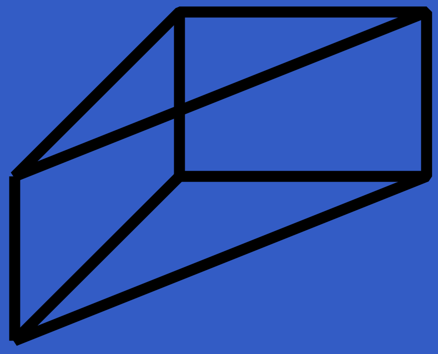
Therefore $I(X_{\Delta_n}) = \langle 0 \rangle$ for all n.

Prisms

The product of two polytopes $P \subseteq \mathbb{R}^m$ and $Q \subseteq \mathbb{R}^n$ is $P \times Q = \{(p,q) \in \mathbb{R}^{m+n} \mid p \in P, q \in Q\}$. The product P×[0,1] is called the prism of P.



P



prism of P

Prisms and Ideals

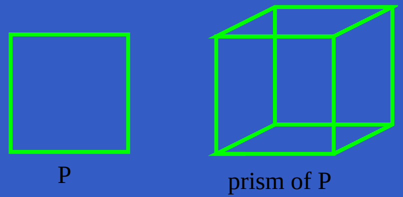
Proposition: Let P be a full-dimensional lattice polytope in $\mathbb{R}^n$. There are two copies of $I(X_P)$ in $I(X_{P\times[0,1]})$.

Example of Proposition

Let P = 2-cube. $P \times [0,1]$ = 3-cube.

$P \cap \mathbb{Z}^2 = \{(0,0),(1,0),(0,1),(1,1)\}$.

$(P \times [0,1]) \cap \mathbb{Z}^3 = \{(0,0,0),(1,0,0),(0,1,0),(1,1,0),$
$(0,0,1),(1,0,1),(0,1,1),(1,1,1)\}$.



$I(X_P) = \langle x_1x_2 - x_0x_3 \rangle$

$I(X_{P\times[0,1]}) = \langle\ y_1y_2 - y_0y_3\ , x_3y_2 - x_2y_3\ , x_1y_2 - x_0y_3\ , x_3y_1 - x_1y_3\ , x_2y_1 - x_0y_3\ , x_3y_0 - x_0y_3\ , x_2y_0 - x_0y_2\ , x_1y_0 - x_0y_1, x_1x_2 - x_0x_3\ \rangle$.

Products

Conjecture: If P and Q are lattice polytopes, $I(X_P) \subset I(X_{P\times Q})$ and $I(X_Q) \subset I(X_{P\times Q})$.

Acknowlegments

- Dr. Jessica Sidman for helping me refine my ideas, notation, and this presentation.
- My fellow REU students: Sarah Gilles, Vince Lyzinski, Aaron Wolbach, and Frances Worek.
- Sarah Gilles for providing the pictures of the n-simplices and n-cubes.
- Macaulay2 for calculating ideals for me.
- Funding for the work presented here comes from NSF DMS-0353700.