Algorithms for Constructing Even-Sided Selfagons

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Mount Holyoke College REU 2005

Studying the Selfatope!

MHC REU 2005 Research Goals:

- study the selfatope, a special type of polytope used in a paper by Jessica Sidman and David Cox
- relate selfatope to toric varieties

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- Def: The *convex hull* of a set A is the intersection of all convex sets containing A; ie, the smallest convex set containing A.

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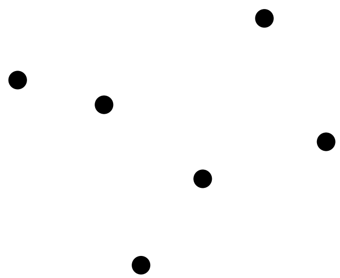
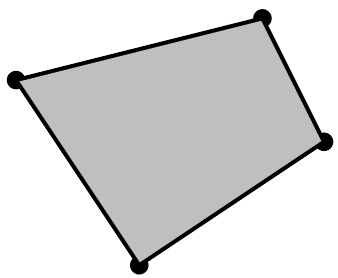
- Ex:



Set of Points



Polytope

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  - Def: *Lattice Polytope* - a polytope with integer valued vertices
  - Def: *Lattice-free Edges* - no lattice points on an edge between vertices

Defining Smooth

- Def: $P \in \mathbb{R}^n$ - lattice polytope
  v vertex in P
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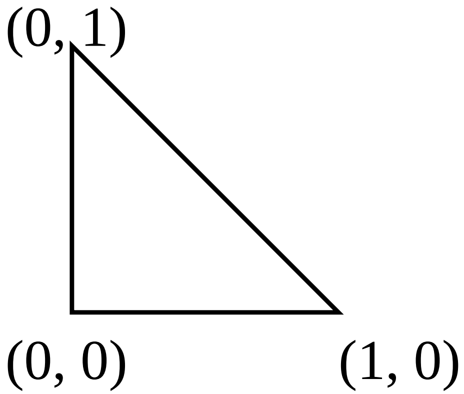
- ie

$$\begin{vmatrix} w_1 - v \\ \vdots \\ w_k - v \end{vmatrix} = \pm 1.$$

Example of Smooth Polytope



$$\begin{vmatrix} (1,0)-(0,0) \\ (0,1)-(0,0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\begin{vmatrix} (0,0)-(1,0) \\ (0,1)-(1,0) \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} = -1$$

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Questions

- Given a particular n, can you construct a selfagon with n edges?

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- Given two selfagons with n edges, are they equivalent?

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  - equivalence to Lyzinski Algorithm

Algorithm 1

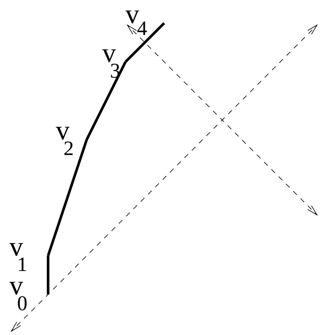
Algorithm for n-gon, $2 \mid n$, $4 \nmid n$

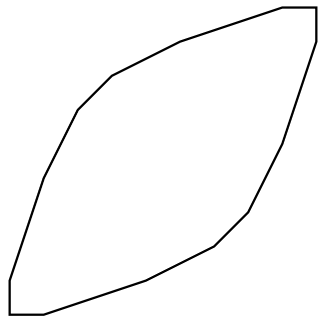
- $v_0 = (0, 0)$
- $v_1 = (0, 1)$
- $v_i = v_{i-1} + (1, \frac{n-2}{4} - i + 2)$, $2 \leq i \leq \frac{n+2}{4}$
- Reflect about $y = x$ and $y = -(x - x_{\frac{n-2}{4}}) + y_{\frac{n+2}{4}}$.

Algorithm 1 - 14-gon

Ex:

- $v_0 = (0, 0)$
- $v_1 = (0, 1)$
  $\frac{n+2}{4} = 4 \Rightarrow$ compute $v_2, v_3,$ and v_4
- $v_2 = v_1 + (1, 3) = (1, 4)$
- $v_3 = v_2 + (1, 2) = (2, 6)$
- $v_4 = v_3 + (1, 1) = (3, 7)$
- Reflect about $y = x$ and $y = -x + 9$.





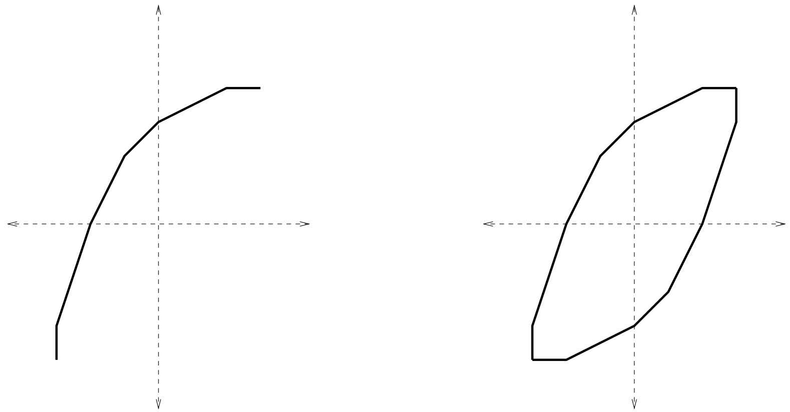
Algorithm 2

Algorithm for n-gon, $4 \mid n$

- $v_0 = (0, 0)$
- $v_1 = (0, 1)$
- $v_i = v_{i-1} + (1, \frac{n}{4} + 2 - i)$, $2 \leq i \leq \frac{n+4}{4}$
- $v_i = v_{i-1} + (i - \frac{n}{4}, 1)$, $\frac{n+8}{4} \leq i \leq \frac{n-2}{2}$
- $v_{\frac{n}{2}} = v_{\frac{n-2}{2}} + (1, 0)$.
- Reflect about $y = \frac{1}{2} y_{\frac{n}{2}}$, $x = \frac{1}{2} x_{\frac{n}{2}}$.

Algorithm 2 - 12-gon

Ex:



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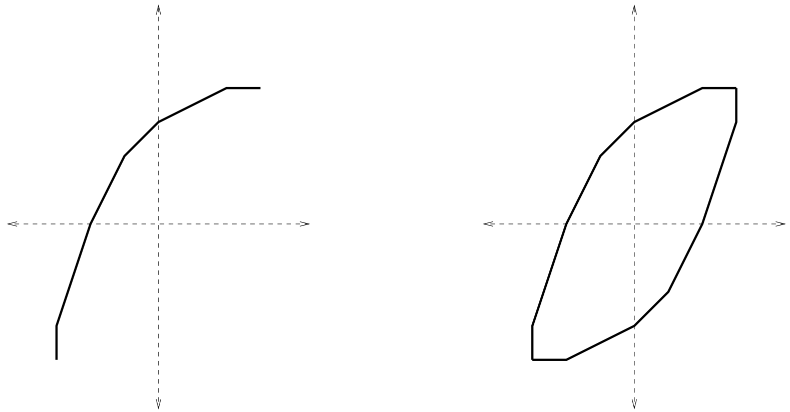

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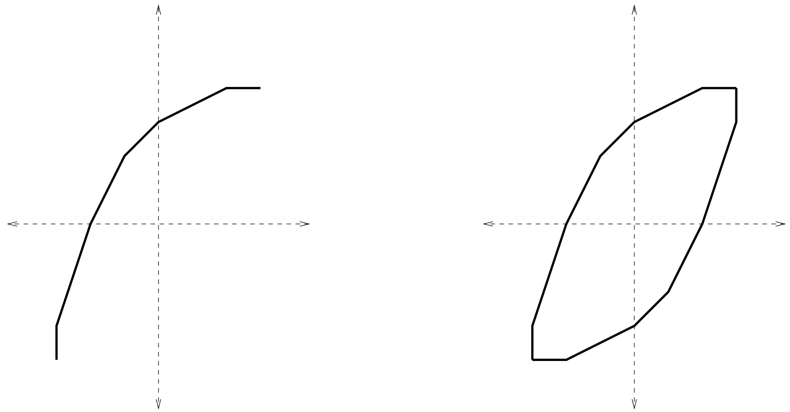

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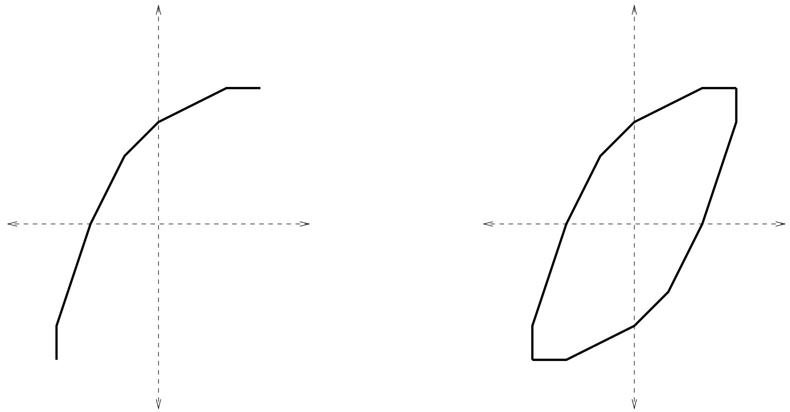

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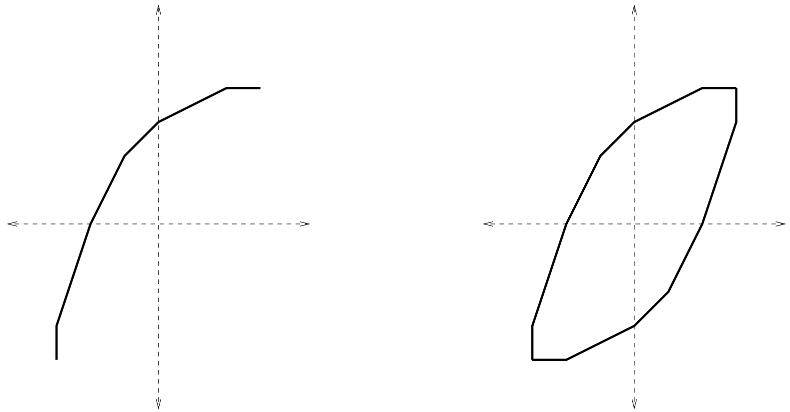

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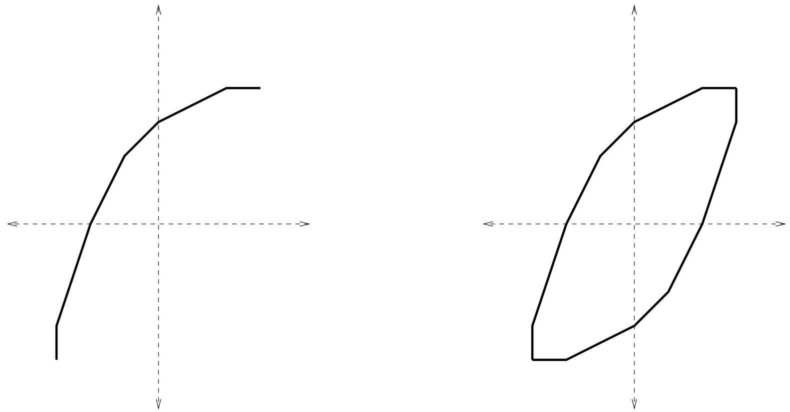

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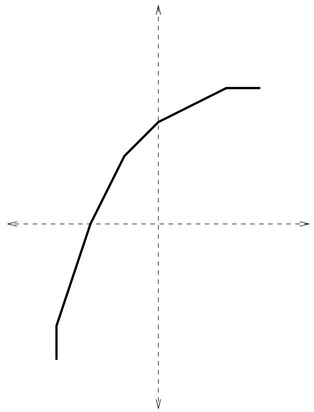

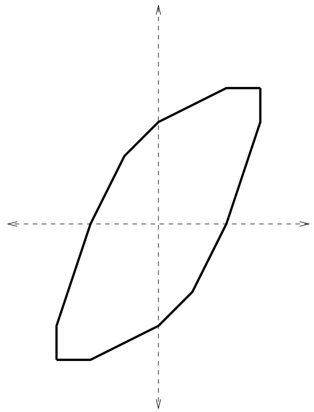

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- $\frac{n-2}{2} = 5 \Rightarrow$ use $v_i = v_{i-1} + (i - 3, 1)$ for v_5

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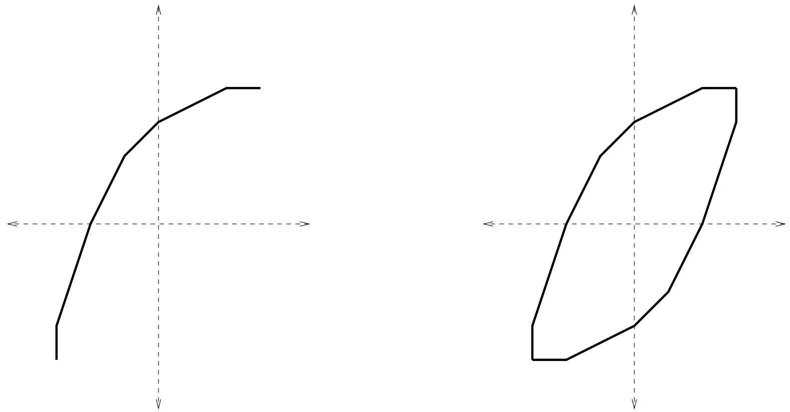

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  - $v_5 = v_4 + (2, 1) = (5, 8)$

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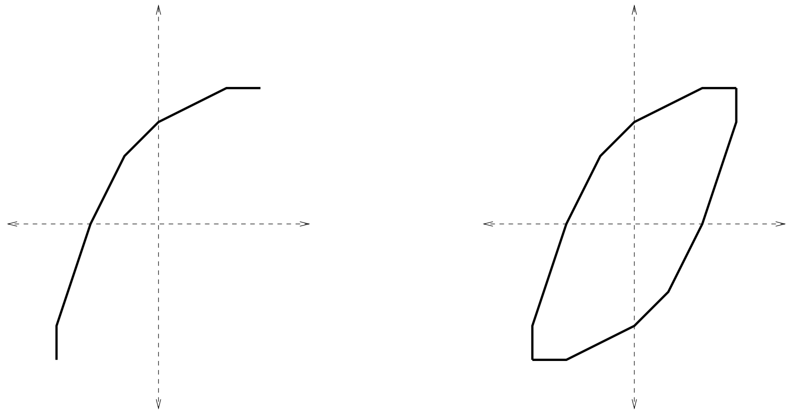

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  - $v_5 = v_4 + (2, 1) = (5, 8)$
- $v_6 = v_5 + (1, 0) = (6, 8)$

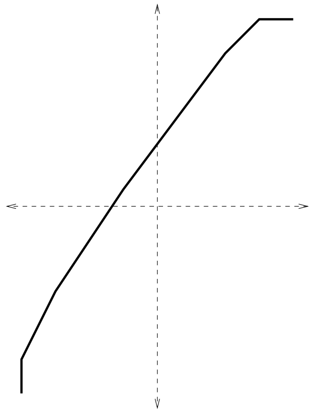
Algorithm 3

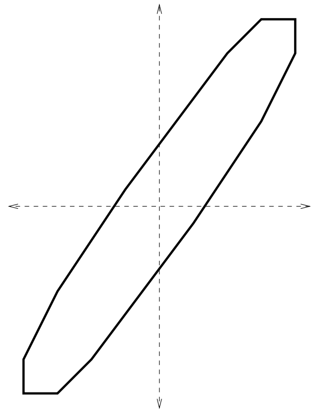
Algorithm for n-gon, n even

- $v_0 = (0, 0)$
- For $i = 1, \ldots, \frac{n-4}{2}$, let $v_i = v_{i-1} + (i-1, i)$.
- $v_{\frac{n-4}{2}+1} = v_{\frac{n-4}{2}} + (1, 1)$.
- $v_{\frac{n-4}{2}+2} = v_{\frac{n-4}{2}+1} + (1, 0)$.
- Reflect about $y = \frac{1}{2} y_{\frac{n}{2}}$, $x = \frac{1}{2} x_{\frac{n}{2}}$.

Algorithm 3 - 12-gon

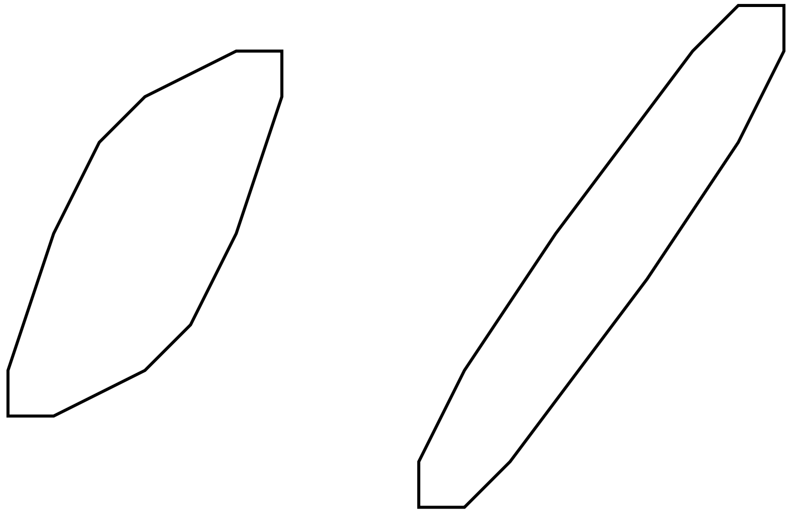
i	v_i
0	(0, 0)
1	(0, 1)
2	(1, 3)
3	(3, 6)
4	(6, 10)
5	(7, 11)
6	(8, 11)





The lines of reflection are $y = \frac{11}{2}, x = 4$.

Comparison of our two 12-gons



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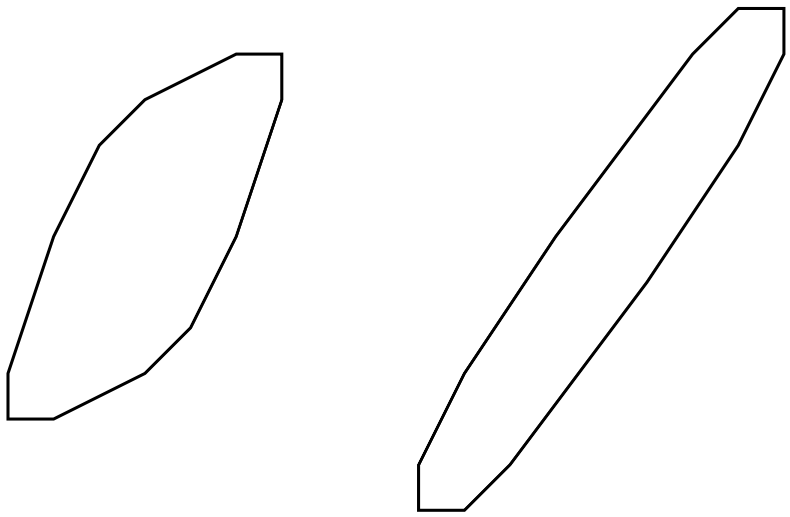

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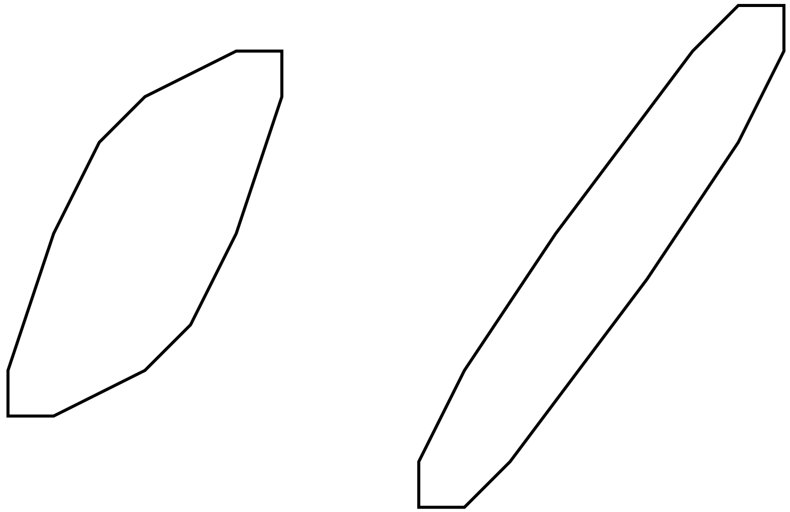

- Equivalent?
  - Def: polytopes P, Q, are *equivalent* if there exists a linear transformation $A \in GL_n(\mathbb{Z})$ such that $AP = Q$
- Not equivalent! - 33 lattice points (Algorithm 2) vs. 32 lattice points (Algorithm 3)

So Algorithm 3 and Algorithm 2 don’t always produce equivalent selfagons.

However, Algorithm 3 and the Lyzinski Algorithm do.

Lyzinski algorithm $4 \mid n$

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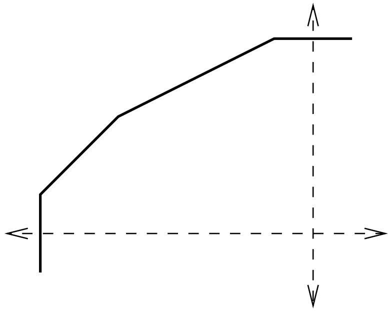
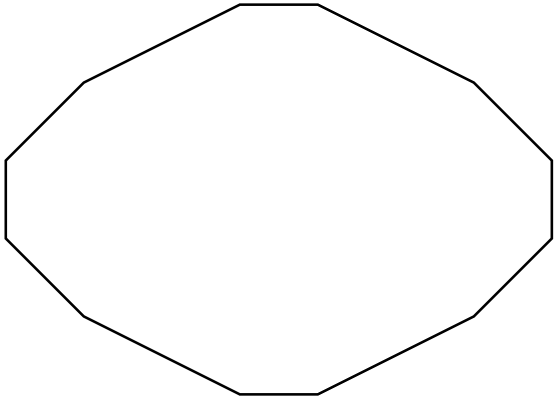
Lyzinski algorithm $4 \mid n$

- $v_0 = (0, 0)$
- $v_i = v_{i-1} + (i - 1, 1),\, 1 \leq i \leq \frac{n}{4}$
- $v_{\frac{n+4}{4}} = v_{\frac{n}{4}}$ + (1, 0)
- Reflect this segment about the line $y = \frac{1}{2}$ and $y = x_{\frac{n+4}{4}} - \frac{1}{2}$.

Lyzinski Algorithm, 12-gon

i	v_i
0	(0, 0)
1	(0, 1)
2	(1, 2)
3	(3, 3)
4	(4, 3)





Lyzinski Algorithm, $2 \mid n$, $4 \nmid n$

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- $v_0 = (0, 0)$
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- $v_j = v_{j-1} + (\frac{n}{2} + 1 - j, -1)$, $\frac{n+10}{4} \leq j \leq \frac{n}{2}$

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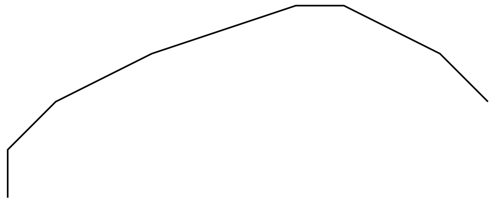
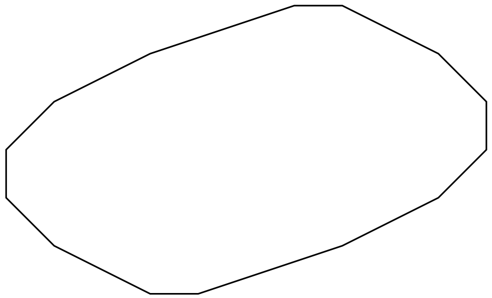
- $v_j = v_{j-1} + (\frac{n}{2} + 1 - j, -1)$, $\frac{n+10}{4} \leq j \leq \frac{n}{2}$
- Rotate this segment to complete the selfagon.

Lyzinski Algorithm, 14-gon

i	v_i
0	(0, 0)
1	(0, 1)
2	(1, 2)
3	(3, 3)
4	(6, 4)
5	(7, 4)
6	(9, 3)
7	(10, 2)





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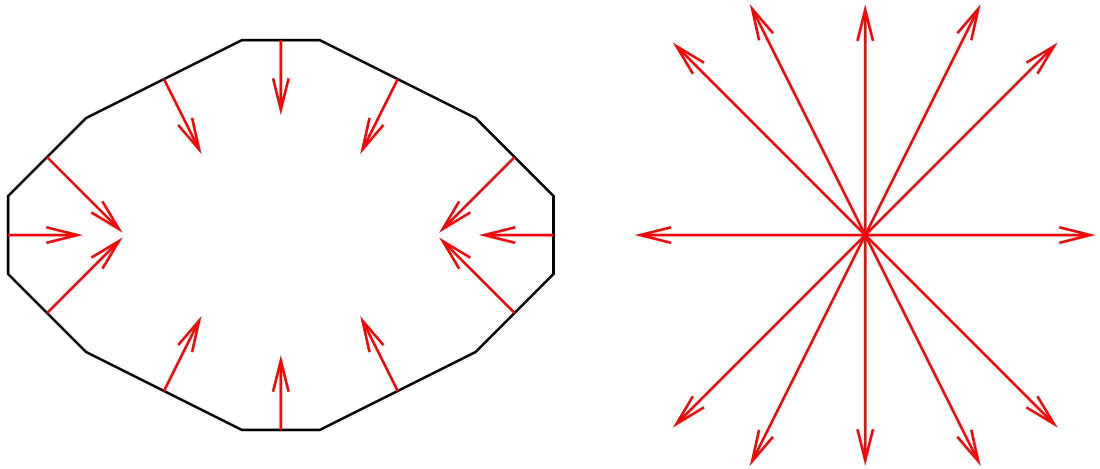
Sketch of proof:

- want to find linear trans. A s.t. $AP = Q$
- problem - to which vertices in Q can we map a given vertex in P and find this A?
- use inner normal fans to solve this problem

Inner Normal Fan

Example of Inner normal fan of polytope:



Thm (Fulton)-

r_i ray of inner normal fan

$a_i r_i = r_{i-1} + r_{i+1}$,

a_i constant, invariant under linear transformation

Equivalence of Algorithms

Sketch of proof, ctd:

- Consider inner normal fans of the two selfagons.

Equivalence of Algorithms

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  - Determine linear transformation B between fans, using the invariants a_i to determine which rays in the first fan map to which rays in the second fan.

Equivalence of Algorithms

Sketch of proof, ctd:

- Consider inner normal fans of the two selfagons.
  - Determine linear transformation B between fans, using the invariants a_i to determine which rays in the first fan map to which rays in the second fan.
  - Use that $A = (B^{-1})^T$ to find A, the transformation between the selfagons.

Equivalence of Algorithms

We find that transformations between P generated by Lyzinski Alg. and Q generated by 3rd Alg is:

- For n-gons with $4 \mid n$, $A = \begin{pmatrix} 1 & \frac{n-8}{4} \\ 1 & \frac{n-4}{4} \end{pmatrix}$.

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- For n-gons with $4 \mid n$, $A = \begin{pmatrix} 1 & \frac{n-8}{4} \\ 1 & \frac{n-4}{4} \end{pmatrix}$.
- For n-gons with $2 \mid n$, $4 \nmid n$,
  $A = \begin{pmatrix} -1 & -\frac{n-10}{4} \\ -1 & -\frac{n-6}{4} \end{pmatrix}$.

Acknowledgements

Thank you to:

- Dr. Jessica Sidman
- Dr. Margaret Robinson
- Fellow REU participants, especially Joanna Miles for her help with *Prosper*
- NSF, for funding this REU