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Project Description

The goals for this summer's work were twofold. In 2006, my REU group generated a large amount of data on class numbers associated to binary quadratic forms with given discriminant d, not a perfect square. In fact, they had computed such class numbers up to d = 52 million and fundamental units of associated orders in quadratic number fields up to d = 750 million. A famous outstanding conjecture of Gauss states that the class number is one for infinitely many positive discriminants. Genus theory, which he developed to examine this question, restricts the set of possible such discriminants according to their factorability. Any fundamental discriminant with many odd prime factors must necessarily have class number larger than one. This year, we used the data generated earlier to examine asymptotics of class numbers along the sequences of fundamental discriminants allowable by genus theory. Because the data set was so large, we were not able to finish a detailed analysis of our results for each sequence, but we expect to have that done in time for the undergraduate presentations at the January meetings.

The second topic we examined involved "relative class numbers", associated to class numbers of orders in quadratic fields with field discriminants d_0 or $4d_0$, for d_0 square - free. The integers in such number fields are of the form $Z + Z\omega_{d_0}$, where

$$\omega_{d_0} = \begin{cases} \sqrt{d_0} & \text{if } d_0 \equiv 2,\ 3 \pmod 4 \\ \dfrac{1+\sqrt{d_0}}{2} & \text{if } d_0 \equiv 1 \pmod 4. \end{cases}$$

Orders in such fields are rings of the form $Z + Zf\omega_{d_0}$, and correspond to discriminants $d = f^2 d_0$ of binary quadratic forms. Dirichlet showed that for $d_0 = 2$ *and* 5, an infinite set of orders have $h(f^2 d_0) = h(d_0)$. The relative class number is the quotient

$$h(f^2 d_0)/h(d_0),$$

and it is not known if for every d_0 there exists an f for which this ratio is one. We looked at this question, which turns out to be equivalent to asking whether one can characterize the set of discriminants d_0 that divide the y-coefficient of the fundamental unit of the quadratic field, $Q(\sqrt{d_0})$. We discovered that this is a condition important to conjectures arising from investigations into an outstanding question posed by Erdos on the existence of three consecutive "powerful" integers. We were analyzing an algorithm proposed by Stephens and Williams (1988) which may give a way to construct such discriminants. If so, we will be able to answer our initial question in the negative. In addition to this, we examined asymptotics that describe the growth of the relative class number for fixed d_0 and increasing f. All final results will be posted on our REU web page.