Introduction

You undoubtedly have been told that the best way to learn mathematics is to do mathematics. But what does it mean to *do mathematics*? To most people, doing mathematics is working out discrete problems in textbooks. But to those who love mathematics, doing mathematics is exploring mathematical phenomena. It is investigating, discovering, being mystified, and finally (one hopes) understanding. It is finding an illuminating way to think about something that then leads to more questions and more possibilities, and delighting in the way in which elementary notions turn out to be unexpectedly subtle and seemingly intractable difficulties sometimes yield to a new idea.

This book aims to lead you into the *doing* of mathematics. It consists of a series of laboratory projects, most of which use the computer as an experimental tool. For each project, your tasks will be

- To work by hand and/or by computer to generate *examples* illuminating questions asked—and to raise some *questions* of your own;
- To carry out suitable experiments to enable you to see *patterns* in the data relating to the problem under investigation;
- To give clear descriptions of your *experimental findings*;
- To make *conjectures* based on your observations;
- To support your conjectures with arguments based on your empirical evidence, on mathematical analysis, and—when possible—with *mathematical proofs*.

The topics range across many different areas of mathematics and statistics. We have chosen them both to convey some of the breadth of the mathematical sciences and also to introduce you to a number of important ideas that you will encounter again in future courses.

With a few exceptions described below, the chapters (projects) can be covered in any order. You can have a valuable learning experience working through just one or two, or half a dozen or more.

The goal is for you to have the fun of discovering some mathematics on your own. While you will learn some specific ideas and techniques, these are secondary to the broader experience of mathematical inquiry. Of course, what distinguishes mathematics from other scientific disciplines is the possibility that empirical findings can be rigorously *proved* beyond a shadow of a doubt. You won't always know enough mathematics to construct such a proof, but you can create a good foundation for learning to construct proofs in more advanced courses by formulating your empirical arguments rigorously and clearly.

Each chapter introduces a topic and places it in some context, often with exercises to give you practice with new ideas. It then raises a number of more substantial questions for you to investigate, with suggestions for how to get started. The chapter concludes with a discussion of some of the underlying mathematical ideas, to help you understand and interpret your results and to give you ideas for how to support some of your conjectures with analysis or, in some cases, proof.

You will find it useful to keep a laboratory notebook of all your experiments, jotting things down as you do them and recording your observations and guesses. Your notebook can be the basis for discussion with fellow students as well as for writing a report summarizing the results of your investigation.

Writing a report is an invaluable opportunity to clarify and refine your thinking. Your instructor may specify which questions your report should address, or you may choose a cluster of related questions that interest you. We suggest that you write your report so that it makes sense to a reader who has taken a semester of college level mathematics but has not worked with this material. (If you have a friend who fits this description and is willing to read and comment on your drafts, you have a treasure!) You should write in full sentences and paragraphs—no cryptic strings of formulas. Try to be both clear and interesting. Look at a math text you particularly liked or an article you enjoyed reading to get an idea of a tone and style to aim for.

Your introduction should describe the topic under investigation in a way that engages the reader's interest. You may need to provide some background or context for your investigation. Define with care the terminology that you will use, since precise descriptions of the phenomena you observe are essential. (Often it is easiest to write the introduction last!) The body of your report naturally falls into four sections:

1. Your experimental strategy or *design.*

   You should motivate the questions you ask—and the order in which you ask them—and explain the logic of your choice of examples. Here are some specific suggestions to get you started.

   (a) Describe your first example.

   - What was it?
   - Describe it geometrically and/or algebraically.
   - Why did you choose it?
   - What happened when you carried it out?

   (b) What did you try next? Why? What were your results?

   (c) What eventually evolved as your general strategy for choosing examples? Why?

2. Results of your experimentation.

   You should organize your data carefully and give thought to how you display your results; make effective use of tables, graphs, and pictures.

   (a) Describe how your various examples worked out, being as clear as you can, but omitting details that don't seem important.

   (b) Attach tables or graphs or sketches to your description, where appropriate. Give each a clear, informative title. However, don't include anything you don't refer to in your discussion section.

3. Analysis of data.

   Organize the discussion of your data carefully, and refer to your results by citing the titles and numbers you assign. (E.g., a report on Chapter 4 might refer to "Table 3, Mersenne primes.") Explain how your data support your conjectures.

   (a) What patterns do you observe in your data?

(b) Formulate your conjectures. Which patterns do you guess represent *real* phenomena, rather than accidental regularities of the examples you happened to choose?

(c) Justify your conjectures. Your choice of examples should stringently test your conjectures—try to rule out "chance" regularities.

4. Mathematical analysis of conjectures.

   Back up your empirical argument with an analytical and/or theoretical one when you can.

You will need access to a computer, of course. Most chapters require one or more simple computer programs. Programs are described in the text in *pseudocode*, an outline of the program that makes its logic clear without burdening the reader with the details of the syntax of any particular programming language. Chapter 1 includes an introduction to pseudocode and the logic of a typical computer algorithm called a FOR-NEXT loop. Working code for each program is provided at the end of the chapter in two languages, *True*BASIC and *Mathcad*. Three of the chapters (5, 11, and 12) use more complicated programs, available electronically and on disk. Details are in the instructor's manual accompanying this text. We do not assume prior knowledge of programming, and learning to program is not one of the goals of this text. However, by trying to *read* the programs and understand how they work you will learn something about what computers can do and how to make them do it.

Most of the chapters are completely independent of each other and of specific prior courses, but there are some exceptions. Chapters 13, 14, and 15, Iteration to Solve Equations, Iteration of Quadratic Functions, and Iteration of Linear Maps in the Plane—are independent of each other, but they each assume that you have worked through Chapter 1, Iteration of Linear Functions. Chapter 16, the Euclidean Algorithm for the Complex Integers, assumes you have done Chapter 3 on the Euclidean algorithm for ordinary integers. Also, the introduction to *modular arithmetic* in Chapter 2 (see Section 2.2) is helpful, but not essential, for Chapters 3, 4, and 8.

Most of this text doesn't require calculus. The derivative appears briefly in portions of Chapters 13 and 14, but these portions can be avoided. There is modest use of the definite integral and the fundamental theorem of calculus in Chapter 10, Numerical Integration, and Chapter 11, Sequences and Series. However, the investigations in Chapter 10 are mostly self-contained and actually provide a good introduction to the integral. Integration and the fundamental theorem are used more heavily in Chapter 12, Experiments in Periodicity.

We hope you enjoy exploring these projects and that they whet your appetite for further study.