

Suppose you have a function $y(x)$ and you want to “create” it out of two other (linearly independent) functions $y_1(x)$ and $y_2(x)$ using constants c_1 and c_2 as follows:

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

It is clear that in general it will be impossible to pick optimal values of c_1 and c_2 for *all* values of x . However, at a specific point x_0 we can write

$$y(x_0) = c_1 y_1(x_0) + c_2 y_2(x_0) \quad (1)$$

In addition, since we need another equation for c_1 and c_2 , we should also require

$$y'(x_0) = c_1 y_1'(x_0) + c_2 y_2'(x_0) \quad (2)$$

so we pick constants that match both the function and its slope at the point x_0 .

The values we get for c_1 and c_2 by simultaneously solving equations (1) and (2) will give us the best 2 parameter approximation for $y(x)$ since it gives exact values for $y(x_0)$ and $y'(x_0)$ and

$$y(x) \approx y(x_0) + \left. \frac{dy}{dx} \right|_{x=x_0} (x - x_0)$$

To reconstruct the *entire* function $y(x)$ we replace c_1 and c_2 by $v_1(x)$ and $v_2(x)$ where v_1 and v_2 are functions which give optimal values of $y(x)$ for *all* x

$$y(x) = v_1(x) y_1(x) + v_2(x) y_2(x) \quad (3)$$

and

$$y'(x) = v_1(x) y_1'(x) + v_2(x) y_2'(x) \quad (4)$$

for the equations which must be satisfied for $v_1(x)$ and $v_2(x)$.

By differentiating equation (3) we get

$$y'(x) = v_1'(x) y_1(x) + v_1(x) y_1'(x) + v_2'(x) y_2(x) + v_2(x) y_2'(x) \quad (5)$$

To make equation (5) consistent with equation (4) we require

$$v_1(x) y_1'(x) + v_2(x) y_2'(x) = v_1'(x) y_1(x) + v_1(x) y_1'(x) + v_2'(x) y_2(x) + v_2(x) y_2'(x)$$

or

$$v_1'(x) y_1(x) + v_2'(x) y_2(x) = 0 \quad (6)$$

Now if we have a linear, non-homogeneous, second order differential equation with constant coefficients

$$y''(x) + 2ay'(x) + b^2y = g(x)$$

with solutions to the homogeneous equation

$$y_h(x) = c_1y_1(x) + c_2y_2(x)$$

we try to find the particular solution

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x) \tag{7}$$

The best approximation we can make to $y_p(x)$ is when

$$y_p'(x) = v_1(x)y_1'(x) + v_2(x)y_2'(x)$$

or

$$v_1'(x)y_1(x) + v_2'(x)y_2(x) = 0 \tag{8}$$

In addition, $y_p(x)$ has to satisfy the differential equation

$$y_p'' + 2ay_p' + b^2y_p = g(x)$$

Substituting equation (7) into this equation and using the condition given by equation (8) gives the results we obtained in class. (This process can be extended to higher order linear differential equations with constant coefficients where the extra solutions allow the matching the expression for the particular solution to higher order derivatives of $y_p(x)$.)