A Solution to $y''-y=x+1;\ y(0)=y'(0)=0$ by using power series

Since the initial conditions are for x=0, use a MacLaurin series to represent the unknown function y: $y=\sum_{n=0}^{\infty} c_n x^n$, $y'=\sum_{n=1}^{\infty} nc_n x^{n-1}$, and $y''=\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$. Since $y(0)=y'(0)=0$, $c_0=c_1=0$. Equating the coefficients for the constant, linear, quadratic, … terms in the equation $y''-y=x+1$, we get:

Constant Terms: $2c_2 - c_0 = 1$ (It may help to write out the two power series.)
Coefficients of Linear Terms: $(3)(2)c_3 - c_1 = 1$
Coefficients of Quadratic Terms: $(4)(3)c_4 - c_2 = 0$
Coefficients of Cubic Terms: $(5)(4)c_5 - c_3 = 0$
Coefficients of Degree 4 Terms: $(6)(5)c_6 - c_4 = 0$
…
Coefficients of Degree n (n>3) Terms: $(n)(n-1)c_n - c_{n-2} = 0$

Then, using the values $c_0=c_1=0$, we get: Solving the constant-term equation $c_2 = \frac{1}{2}$
Solving the linear-term equation $c_3 = \frac{1}{(3)(2)}$
Solving the quadratic-term equation $c_4 = \frac{1}{(4)(3)(2)}$
Solving the cubic-term equation $c_5 = \frac{1}{(5)(4)(3)(2)}$
… and the pattern is pretty clear, so that the series for y is: $y=\sum_{n=2}^{\infty} \frac{1}{n!} x^n$, which should look very familiar; it's just the Maclaurin series representation for the exponential function e^x - except that it's <u>missing</u> the first two terms, 1 + x, !

What to do? Just add those two <u>in</u> to make the sum equal to e^x … and then, of course, subtract them <u>out</u> to keep the function unchanged: $y=\left(\sum_{n=0}^{\infty} \frac{1}{n!} x^n\right) - (1+x) = e^x - (1+x)$.

Once we have this more "usual" representation of y: $y = e^x - (1+x)$ it is very straightforward to verify that this function is a solution to the differential equation and that it satisfies the initial conditions. So, although the methods are on somewhat shaky ground as far as mathematical rigor goes, there is no doubt that we've gotten a valid solution. The only unverified piece remaining is that the solution is the only one, that it is the unique solution, to the initial-value problem.