Some problems for Potential Mathematics Majors/Minors

1. Suppose that we let H_n represent the nth partial sum of the Harmonic series, $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ ... \ + \frac{1}{n}$. Now use the nth partial sums of the Harmonic series to construct a new infinite series, $\sum_{n=1}^{\infty} \frac{H_{n+1}}{n(n+1)} = \frac{H_2}{2} + \frac{H_3}{6} + \frac{H_4}{12} + \frac{H_5}{20} + \ ...$
Show that this new infinite series converges and determine the number to which it converges. (Hint: One way to do this problem is to apply the method of Partial Fractions, which is described in 7.4, pp 497-498. Try applying this method to $\frac{H_{n+1}}{n(n+1)}$, so that each term in the infinite sum is itself represented as a sum of two terms.)

2. We've described a folding process that we can repeat to produces a sequence that converges to $\frac{1}{7}$. Create a similar process for $\frac{1}{3}$. By similar, I mean that each fold is just a fold of the left endpoint (zero) or the right endpoint (one) onto the previously constructed point. For example, the process for $\frac{1}{7}$ is a repeating of: Fold in half from the right, fold in half from the left, fold in half from the left.

   Only the results of each entire three-step folding process is recorded as part of the sequence, and the initial point is an arbitrary number between 0 and 1. (Suggestion: To invent a process for $\frac{1}{3}$ recall that the folding process for $\frac{1}{7}$ was invented by assuming that the initial point is actually $\frac{1}{7}$, and try to invent folds in half using some combination of folds in half "from the right" and "from the left. Once you invent the process, apply it to an arbitrary number and show that the sequence of points converges to $\frac{1}{3}$. Hint: The process is just a bit simpler than the one for $\frac{1}{7}$.)

3. We determined that the total overhang for the box sliding problem when the first four boxes slid out is $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8}$, and we developed a method for determining the center of gravity for n+1 boxes given the result for n boxes. Express the result for n boxes as an explicit formula in terms of n. Now do the following problem: Suppose that you started sliding boxes at the time of the "Big Bang," sliding one box every second up to today. How many box lengths would the top box overhang at this point? You may assume that the "Big Bang" was

exactly 12 billion years (365 day years) ago. Ideally you will make an estimate without a calculator.

4. This problem outlines a way to find the sum of the alternating Harmonic series. Start with notation for the Harmonic and alternating Harmonic series:

Let h_n denote the nth partial sum of the harmonic series:

$$h_n = \sum_{i=1}^{n} \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ \dots \ + \frac{1}{n}.$$

Let s_n denote the nth partial sum of the alternating harmonic series:

$$s_n = \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ \dots \ + \frac{(-1)^{n+1}}{n}$$

   a. Write the difference between the two partial sums for an <u>even</u> number, 2n, of terms: $h_{2n} - s_{2n}$
   b. Simplify the expression in part a, expressing the result as a partial sum of the Harmonic series.
   c. Solve the equation you got in part b for s_{2n} and simplify, expressing s_{2n} as a sum of just n terms.
   d. Show that s_{2n} is the sum in the Pop Quiz” problem.
   e. Determine $\lim_{n\to\infty} s_{2n}$ using the result of the Pop Quiz” problem.
   f. Express s_{2n+1} in terms of s_{2n} and determine $\lim_{n\to\infty} s_{2n+1}$.
   g. Determine the sum of the alternating Harmonic series.

5. This problem illustrates a way to create a variation on an alternating series. Start with a recursively defined sequence b_n: $b_1 = 1, b_2 = -1, \text{ and } b_n = \frac{b_{n-1}}{b_{n-2}} \text{ for } n > 2$.

   a. Write the first ten terms of the sequence b_n.

   Now consider a sequence c_n defined in terms of b_n: $c_n = \frac{1}{2}b_n - b_{n+1} + \frac{1}{2}b_{n+2}$.

   b. Write the first ten terms of the sequence c_n.

   Finally, consider a sequence a_n defined by: $a_n = \frac{c_n}{n}$

   c. Write the first ten terms of the sequence a_n.
   d. Determine the sum of the infinite series $\sum_{n=1}^{\infty} a_n$ by following the pattern in problem 4 above.