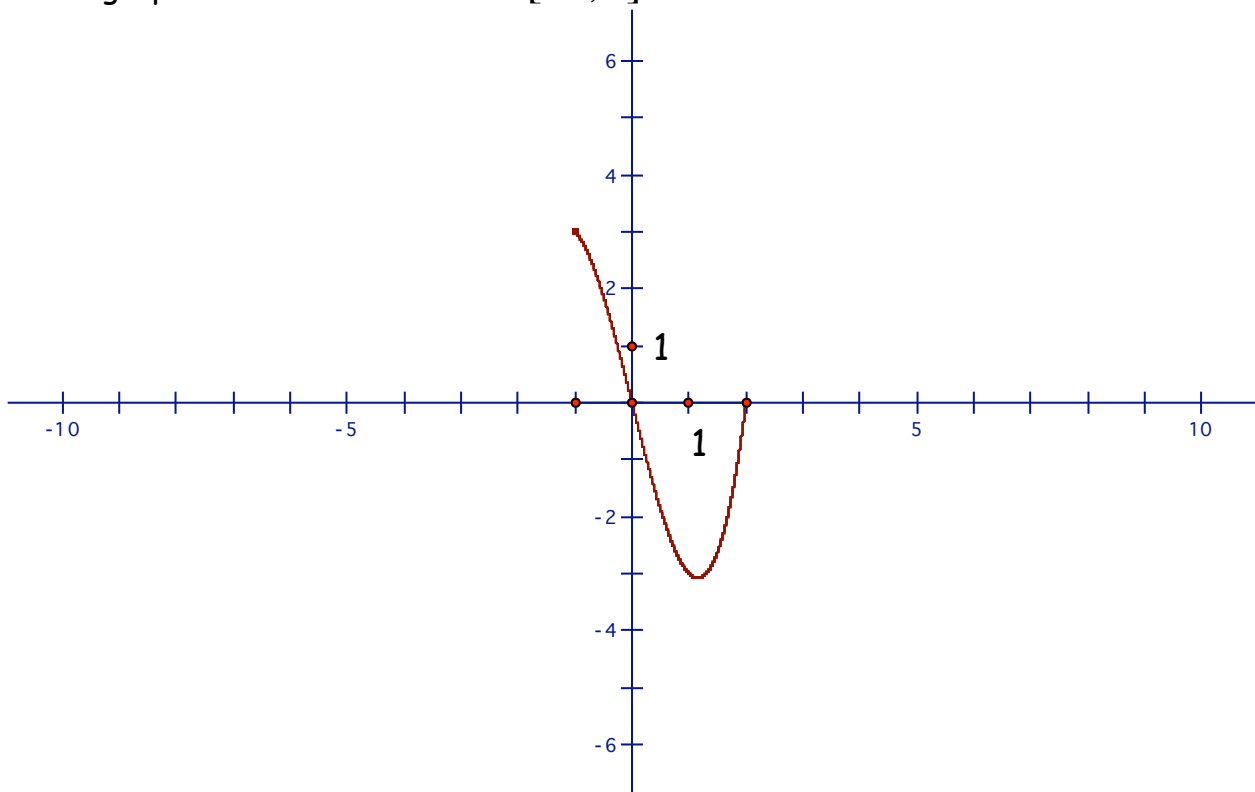


Integral Example
Math 101 - 01

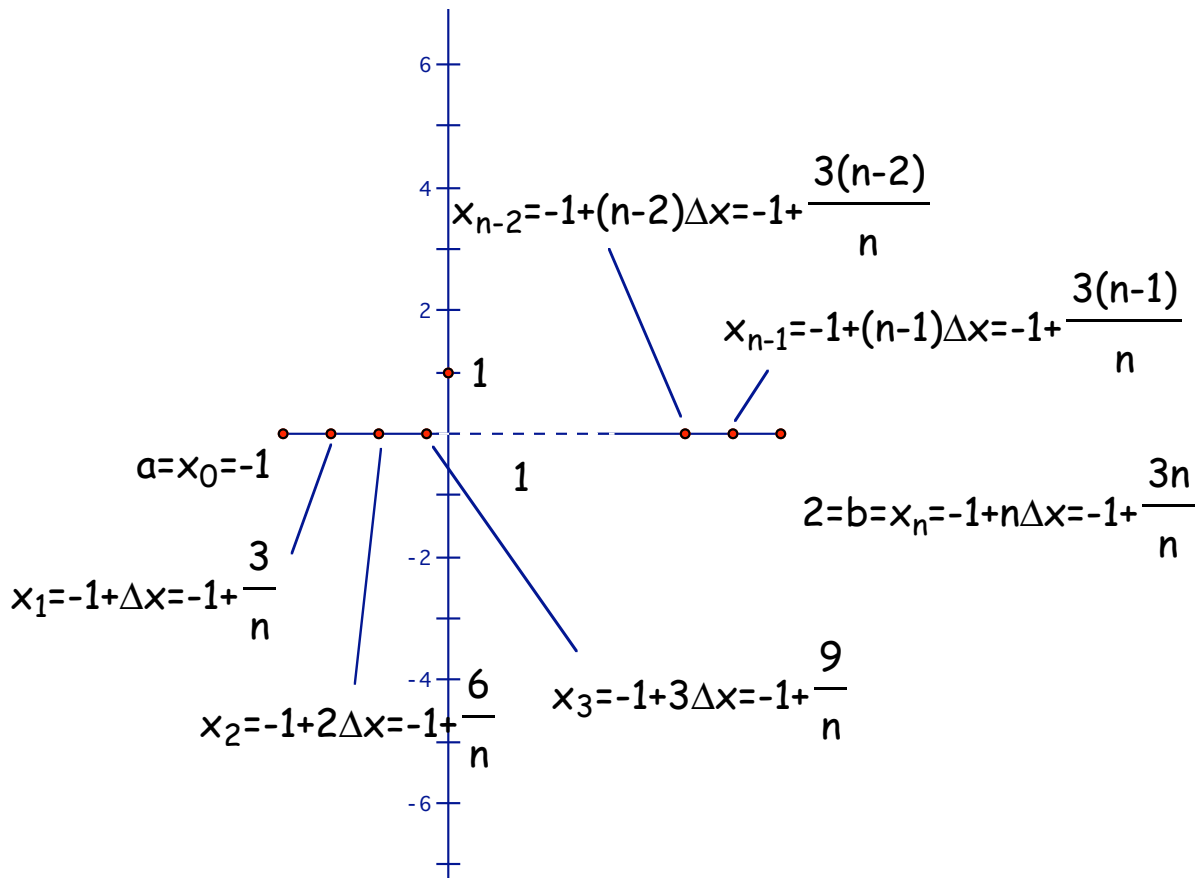
The example below illustrates the use of Theorem 4, p 264, and how to construct a specific form of Riemann Sum whose limit is the definite integral.

Here's the integral to evaluate: The integral of the function f , defined by $f(x) = x^3 - 4x$, over the interval $[-1, 2]$.

1. The first thing is to get a rough picture of the function and interval: The graph of f over the interval $[-1, 2]$ looks like:



2. The integral is a limit of Riemann sums, which depend on a partition of the interval over which the integral is taken. Guided by the requirements of Theorem 4, the interval $[-1, 2]$ is divided into n equal subintervals, each of width $\frac{2 - (-1)}{n} = \frac{3}{n}$. Using the notation of Theorem 4, $\Delta x = \frac{3}{n}$:



3. Again, guided by the requirements of Theorem 4, sample

points: x_1^* , x_2^* , x_3^* , ..., x_{n-2}^* , x_{n-1}^* , x_n^* are chosen to be right endpoints of each of

the subintervals: $x_1^* = x_1 = -1 + \frac{3}{n}$, $x_2^* = x_2 = -1 + \frac{6}{n}$, $x_3^* = x_3 = -1 + \frac{9}{n}$, ... ,

$x_i^* = x_i = -1 + \frac{3i}{n}$, ... , $x_n^* = x_n = -1 + \frac{3n}{n} = 2$

4. Now the Riemann Sum can be formed for this regular partition into n

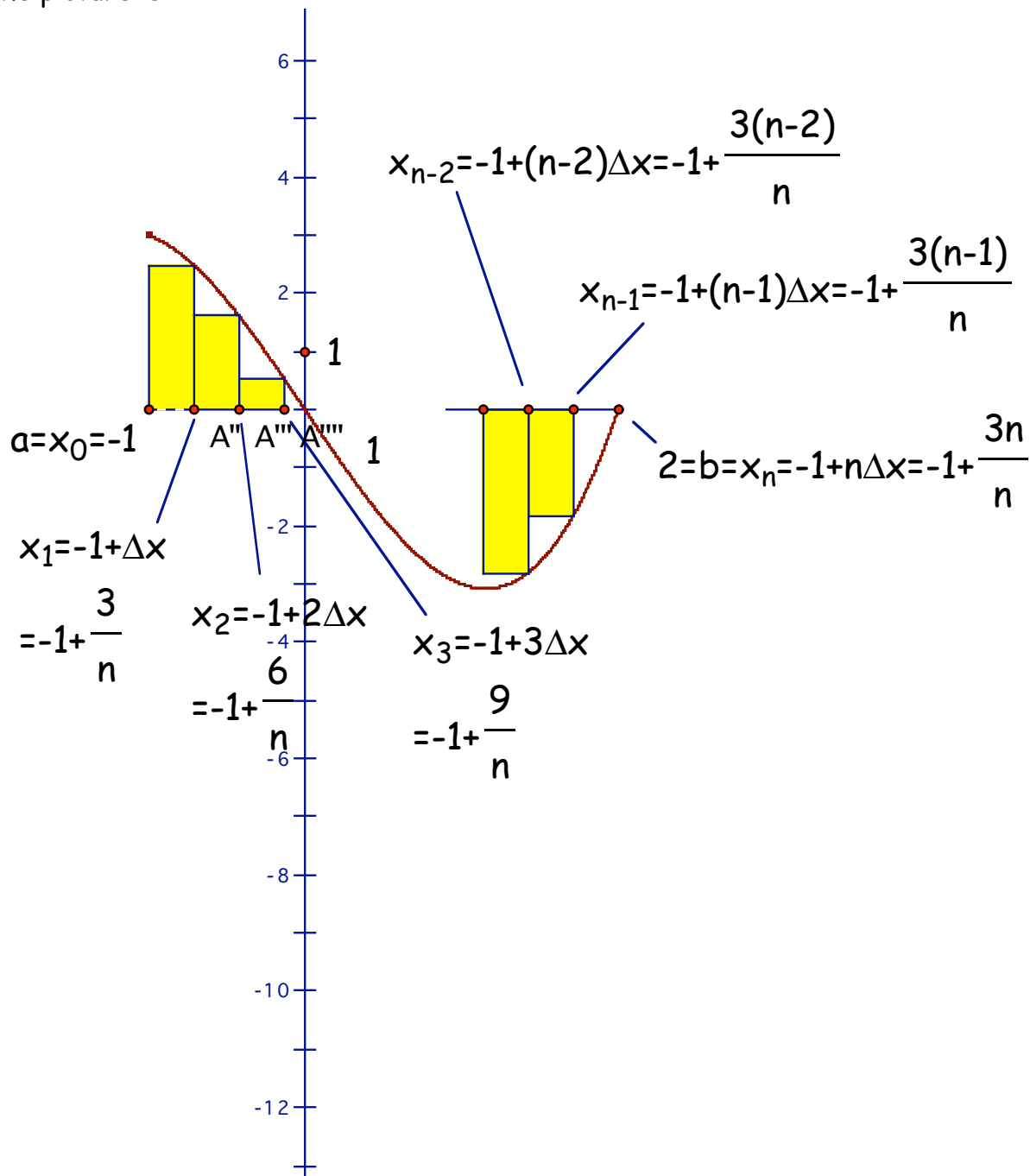
subintervals with sample points the right endpoints of the subintervals: For now in this example, I'll avoid using the sigma notation.)

$f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + \dots + f(x_i^*)\Delta x + \dots + f(x_n^*)\Delta x$, which,

putting in the value of Δx and the x_i^* values, is:

$f(-1 + \frac{3}{n})(\frac{3}{n}) + f(-1 + \frac{6}{n})(\frac{3}{n}) + \dots + f(-1 + \frac{3i}{n})(\frac{3}{n}) + \dots + f(-1 + \frac{3n}{n})(\frac{3}{n})$.

The picture is:



where the Riemann Sum is the sum of the "areas," area being understood to be negative when the graph is below the x-axis.

Now for the algebra!

Factoring out $\frac{3}{n}$ and evaluating the function produces the form

$$\begin{aligned} & \left(\frac{3}{n}\right)\left\{\left(-1+\frac{3}{n}\right)^3-4\left(-1+\frac{3}{n}\right)+\left(-1+\frac{6}{n}\right)^3-4\left(-1+\frac{6}{n}\right)+\dots\right. \\ & \left.+\left(-1+\frac{3i}{n}\right)^3-4\left(-1+\frac{3i}{n}\right)+\dots+\left(-1+\frac{3n}{n}\right)^3-4\left(-1+\frac{3n}{n}\right)\right\}= \\ & \left(\frac{3}{n}\right)\left\{\left[(-1+\frac{3}{n})^3+(-1+\frac{6}{n})^3+\dots+(-1+\frac{3i}{n})^3+\dots+(-1+\frac{3n}{n})^3\right]-\right. \\ & \left.4\left[(-1+\frac{3}{n})+(-1+\frac{6}{n})+\dots+(-1+\frac{3i}{n})+\dots+(-1+\frac{3n}{n})\right]\right\} \end{aligned}$$

for the Riemann Sum. Let's focus on the parts of this sum:

$$\begin{aligned} & (-1+\frac{3}{n})+(-1+\frac{6}{n})+\dots+(-1+\frac{3i}{n})+\dots+(-1+\frac{3n}{n})=-n+\frac{3}{n}(1+2+\dots+i+\dots+n)= \\ \text{a.} \quad & -n+\frac{3}{n}\left(\frac{n(n+1)}{2}\right)=-n+3\left(\frac{(n+1)}{2}\right)=\frac{(n+3)}{2} \end{aligned}$$

b.

$$\begin{aligned} & (-1+\frac{3}{n})^3+(-1+\frac{6}{n})^3+\dots+(-1+\frac{3i}{n})^3+\dots+(-1+\frac{3n}{n})^3= \\ & (-1+3(\frac{3}{n})-3(\frac{3}{n})^2+(\frac{3}{n})^3)+(-1+3(\frac{6}{n})-3(\frac{6}{n})^2+(\frac{6}{n})^3)+\dots+(-1+3(\frac{3i}{n})-3(\frac{3i}{n})^2+(\frac{3i}{n})^3) \\ & +\dots+(-1+3(\frac{3n}{n})-3(\frac{3n}{n})^2+(\frac{3n}{n})^3)=-n+3(\frac{3}{n})(1+2+\dots+i+\dots+n)-3(\frac{3}{n})^2(1^2+2^2+\dots+i^2+\dots+n^2)+ \\ & (\frac{3}{n})^3(1^3+2^3+\dots+i^3+\dots+n^3) \end{aligned}$$

c. Using the summation formulas:

$$\begin{aligned} 1+2+\dots+i+\dots+n & =\frac{n(n+1)}{2} \\ 1^2+2^2+\dots+i^2+\dots+n^2 & =\frac{n(n+1)(2n+1)}{6} \end{aligned}$$

and $1^3+2^3+\dots+i^3+\dots+n^3 = (\frac{n(n+1)}{2})^2$ the expression in part b becomes

$$\begin{aligned} & -n+3(\frac{3}{n})(1+2+\dots+i+\dots+n)-3(\frac{3}{n})^2(1^2+2^2+\dots+i^2+\dots+n^2)+(\frac{3}{n})^3(1^3+2^3+\dots+i^3+\dots+n^3)= \\ & -n+3(\frac{3}{n})\frac{n(n+1)}{2}-3(\frac{3}{n})^2\frac{n(n+1)(2n+1)}{6}+(\frac{3}{n})^3\left(\frac{n(n+1)}{2}\right)^2= \\ & -n+\frac{9(n+1)}{2}-\frac{27(n+1)(2n+1)}{6n}+(\frac{27(n+1)^2}{4n})=-n+\frac{9(n+1)}{2}\left\{1-\frac{(2n+1)}{n}+\frac{3(n+1)}{2n}\right\}= \\ & -n+\frac{9(n+1)}{2}\left\{\frac{2n-2(2n+1)+3(n+1)}{2n}\right\}=-n+\frac{9(n+1)}{2}\left\{\frac{n+1}{2n}\right\}=-n+\frac{9(n+1)^2}{4n}= \\ & \frac{-4n^2+9(n+1)^2}{4n}=\frac{5n^2+18n+9}{4n} \end{aligned}$$

d. Putting the results of part a and part c into the Riemann Sum:

$$\text{Riemann Sum} = \left(\frac{3}{n}\right)\left\{\left[(-1 + \frac{3}{n})^3 + (-1 + \frac{6}{n})^3 + \dots + (-1 + \frac{3i}{n})^3 + \dots + (-1 + \frac{3n}{n})^3\right] - 4\left[(-1 + \frac{3}{n}) + (-1 + \frac{6}{n}) + \dots + (-1 + \frac{3i}{n}) + \dots + (-1 + \frac{3n}{n})\right]\right\} =$$

$$\left(\frac{3}{n}\right)\left\{\left[\frac{5n^2 + 18n + 9}{4n}\right] - 4\left[\frac{(n+3)}{2}\right]\right\} = \left(\frac{3}{n}\right)\left\{\left[\frac{5n^2 + 18n + 9}{4n}\right] - \left[\frac{8n(n+3)}{4n}\right]\right\} =$$

$$\left(\frac{3}{n}\right)\left[\frac{-3n^2 - 6n + 9}{4n}\right] = \left[\frac{-9n^2 - 18n + 27}{4n^2}\right] = \frac{-9}{4} - \frac{9}{2n} + \frac{27}{4n^2}$$

written as $\frac{-9}{4} - \frac{3}{2}\Delta x + \frac{3}{4}(\Delta x)^2$ by using the relationship $\Delta x = \frac{3}{n}$

5. The last step in determining the value of the integral is to take the limit of the Riemann Sum:

$$\int_{-1}^2 (x^3 - 4x)dx = \lim_{\Delta x \rightarrow 0} \left(\frac{-9}{4} - \frac{3}{2}\Delta x + \frac{3}{4}(\Delta x)^2\right) = \frac{-9}{4}, \text{ or}$$

$$\int_{-1}^2 (x^3 - 4x)dx = \lim_{n \rightarrow \infty} \left(\frac{-9}{4} - \frac{9}{2n} + \frac{27}{4n^2}\right) = \frac{-9}{4}$$

With the integral exactly calculated at this point, I'll show, on the next page, how the Riemann Sum determination would look using the sigma notation:

Riemann Sum =

$$\begin{aligned} \left(\frac{3}{n}\right) \left\{ \sum_{i=1}^n \left[\left(-1 + \frac{3i}{n}\right)^3 - 4 \sum_{i=1}^n \left(-1 + \frac{3i}{n}\right) \right] \right\} &= \left(\frac{3}{n}\right) \left\{ \sum_{i=1}^n \left[(-1)^3 + 3\left(\frac{3i}{n}\right) - 3\left(\frac{3i}{n}\right)^2 + \left(\frac{3i}{n}\right)^3 + 4 - \frac{12i}{n} \right] \right\} = \\ \left(\frac{3}{n}\right) \left\{ \sum_{i=1}^n \left[3 - \left(\frac{3i}{n}\right) - 3\left(\frac{3i}{n}\right)^2 + \left(\frac{3i}{n}\right)^3 \right] \right\} &= \left(\frac{3}{n}\right) \left\{ \sum_{i=1}^n 3 - \sum_{i=1}^n \frac{3i}{n} - \sum_{i=1}^n \frac{27i^2}{n^2} + \sum_{i=1}^n \frac{27i^3}{n^3} \right\} = \\ \left(\frac{3}{n}\right) \left\{ 3 \sum_{i=1}^n 1 - \frac{3}{n} \sum_{i=1}^n i - \frac{27}{n^2} \sum_{i=1}^n i^2 + \frac{27}{n^3} \sum_{i=1}^n i^3 \right\} &= \left(\frac{3}{n}\right) \left\{ 3n - \frac{3n(n+1)}{2} - \frac{27n(n+1)(2n+1)}{n^2 \cdot 6} + \frac{27}{n^3} \left(\frac{n(n+1)}{2}\right)^2 \right\} \\ 9 - \frac{9(n+1)}{2n} - \frac{27(n+1)(2n+1)}{2n^2} + \frac{81(n+1)^2}{4n^2} &= 9 - \frac{9}{2} - \frac{9}{2n} - \frac{27(2n^2+3n+1)}{2n^2} + \frac{81(n^2+2n+1)}{4n^2} = \\ 9 - \frac{9}{2} - \frac{9}{2n} - 27 - \frac{81}{2n} - \frac{27}{2n^2} + \frac{81}{4} + \frac{81}{2n} + \frac{81}{4n^2} &= 9 - \frac{9}{2} - \frac{9}{2n} - 27 - \frac{81}{2n} - \frac{27}{2n^2} + \frac{81}{4} + \frac{81}{2n} + \frac{81}{4n^2} \\ = -\frac{9}{4} - \frac{9}{2n} + \frac{27}{4n^2} \end{aligned}$$

and we're back to the point reached without using the sigma notation.