

## Calculus I-01 Exam 3 Solutions

Write solutions completely for maximum credit. Put all work on separate pages and keep this page for your reference. Each of the four problems is worth 10 points.

1. Differentiate each of the following with respect to  $x$ :

a.  $xe^{-x^2}$

b.  $\frac{\ln(2x+1)}{\sqrt{2x+1}}$

a. By the product rule,  $\frac{d(xe^{-x^2})}{dx} = x \frac{d}{dx} e^{-x^2} + e^{-x^2} \frac{d(x)}{dx}$ , and since

$$\frac{d(x)}{dx} = 1 \text{ and } \frac{d}{dx} e^{-x^2} = e^{-x^2} \left( \frac{d}{dx} (-x^2) \right) = e^{-x^2} (-2x), \text{ we get}$$

$$\frac{d(xe^{-x^2})}{dx} = xe^{-x^2} (-2x) + e^{-x^2} = e^{-x^2} (-2x^2 + 1)$$

b. By the quotient rule,

$$\frac{d}{dx} \frac{\ln(2x+1)}{\sqrt{2x+1}} = \frac{\sqrt{2x+1} \left( \frac{2}{2x+1} \right) - \ln(2x+1) (2) \left( \frac{1}{2} \right) (2x+1)^{-1/2}}{2x+1}$$

$$= \frac{\sqrt{2x+1} \left( \frac{2}{2x+1} \right) - \ln(2x+1) (2) \left( \frac{1}{2} \right) (2x+1)^{-1/2}}{2x+1} =$$

$$\frac{\left( \frac{2}{\sqrt{2x+1}} \right) - \frac{\ln(2x+1)}{\sqrt{2x+1}}}{2x+1} = \left( \frac{2 - \ln(2x+1)}{(2x+1)\sqrt{2x+1}} \right)$$

2. Integrate each of the following with respect to  $x$ :

a.  $\int \frac{\cos(x/2)}{1 + \sin(x/2)} dx$ , making the substitution  $u = 1 + \sin(x/2)$

Let  $u = 1 + \sin(x/2)$ . Then  $du = \frac{1}{2}\cos(x/2)dx$  and  $2du = \cos(x/2)dx$ .

Substituting into the integral,

$$\int \frac{\cos(x/2)}{1 + \sin(x/2)} dx = \int \frac{2}{u} du = 2 \ln|u| + c = 2 \ln|1 + \sin(x/2)| + c$$

To confirm the solution, differentiate the answer using the rule for constant multiples and the chain rule twice to get:

$$\begin{aligned} \frac{d}{dx}(2 \ln|1 + \sin(x/2)|) &= 2 \frac{d}{dx}(\ln|1 + \sin(x/2)|) = \\ 2 \frac{1}{1 + \sin(x/2)} \frac{d}{dx}(1 + \sin(x/2)) &= 2 \frac{1}{1 + \sin(x/2)} \cos(x/2) \frac{d}{dx}(x/2) = \\ 2 \frac{1}{1 + \sin(x/2)} \cos(x/2) (1/2) &= \frac{\cos(x/2)}{1 + \sin(x/2)} \end{aligned}$$

b.  $\int \frac{1}{(2x+1)\ln(2x+1)} dx$ , making the substitution  $u = \ln(2x+1)$

Let  $u = \ln(2x+1)$ . Then  $du = \frac{1}{2x+1}(2)dx$  and  $\frac{1}{2}du = \frac{1}{2x+1}dx$ .

Substituting into the integral,

$$\int \frac{1}{(2x+1)\ln(2x+1)} dx = \int \frac{1}{2} \frac{1}{u} du = \frac{1}{2} \ln|u| + c = \frac{1}{2} \ln|\ln(2x+1)| + c$$

To confirm the solution, differentiate the answer using the rule for constant multiples and the chain rule three times to get:

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{2} \ln|\ln(2x+1)|\right) &= \frac{1}{2} \frac{d}{dx}(\ln|\ln(2x+1)|) = \frac{1}{2} \frac{1}{\ln(2x+1)} \frac{d}{dx}(\ln(2x+1)) = \\ \frac{1}{2} \frac{1}{\ln(2x+1)} \frac{1}{2x+1} \frac{d}{dx}(2x+1) &= \frac{1}{2} \frac{1}{\ln(2x+1)} \frac{1}{2x+1} (2) = \frac{1}{\ln(2x+1)} \frac{1}{2x+1} \end{aligned}$$

3. Suppose that  $f$  is a function defined by  $f(x) = \frac{4}{1+2\ln x}$ , and

assume that  $f$  is 1-1.

a. Determine  $f'(x)$ .

Using the quotient rule, the rule for constants, and the rule for sums we get:

$$f'(x) = \frac{(1+2\ln x)\frac{d}{dx}(4) - 4\frac{d}{dx}(1+2\ln x)}{(1+2\ln x)^2} = \frac{-4(\frac{2}{x})}{(1+2\ln x)^2} = \frac{-8}{x(1+2\ln x)^2}. \text{ Writing } f(x)$$

as  $4(1+2\ln x)^{-1}$ , and using the rule for constant multiples, the rule for sums, and the chain rule also works well.

b. Determine a formula for  $f^{-1}(x)$

Start by writing  $y = \frac{4}{1+2\ln x}$  and solve for  $x$ :

$$y(1+2\ln x) = 4 \Rightarrow y + 2y\ln x = 4 \Rightarrow 2y\ln x = 4 - y \Rightarrow \ln x = \frac{4}{2y} - \frac{y}{2y} = \frac{2}{y} - \frac{1}{2}$$

$$\Rightarrow x = e^{\left(\frac{2}{y} - \frac{1}{2}\right)}, \text{ by using the definition of } \ln.$$

Then replace  $x$  by  $y$  and  $y$  by  $x$  to get:  $f^{-1}(x) = e^{\left(\frac{2}{x} - \frac{1}{2}\right)}$

4. Assume that the equation  $xe^y + y = 1$  defines  $y$  implicitly as a function of  $x$ .

a. Determine the value of  $\frac{dy}{dx}$  at the point  $(1,0)$ .

Differentiating each side of the equation and using various rules of differentiation, we have:

$$xe^y + y = 1 \Rightarrow \frac{d}{dx}(xe^y + y) = \frac{d}{dx}(1) \Rightarrow \frac{d}{dx}(xe^y) + \frac{dy}{dx} = 0$$

$$\Rightarrow x\frac{d}{dx}(e^y) + e^y\frac{d}{dx}(x) + \frac{dy}{dx} = 0 \Rightarrow xe^y\frac{dy}{dx} + e^y + \frac{dy}{dx} = 0$$

$$\Rightarrow (xe^y + 1)\frac{dy}{dx} = -e^y \Rightarrow \frac{dy}{dx} = \frac{-e^y}{(xe^y + 1)}. \text{ At the point } (1,0), \text{ where } x=1$$

$$\text{and } y=0, \frac{dy}{dx} = \frac{-e^0}{((1)e^0 + 1)} = \frac{-1}{2}.$$

b. Determine where the tangent line to the graph of the equation  $xe^y + y = 1$  at the point  $(1,0)$  intersects the  $y$ -axis. (If you do not get an answer to part a, state an assumption for the answer to part a in order to answer this part of the problem.)

Since the tangent line passes through the point  $(1,0)$ , if you go to the left 1 unit from  $(1,0)$  to get to the  $y$ -axis, you would need then to go up  $1/2$  unit to get to a point on the  $y$ -axis that is on the tangent line, since the slope of the tangent line is  $-1/2$ .