Solutions to some Modelling Problems

(p. 556, #12) A mothball is in the shape of a sphere and starts with radius 1 cm. The material in the mothball evaporates at a rate proportional to the surface area. After one month the radius is 0.5 cm. How many months (from the start) is it before the radius is 0.2 cm?

Solution: We have to think how evaporation actually happens. Molecules leave the mothball, and each molecule takes up some volume in the solid. It makes sense that only molecules on the surface could leave the mothball, and hence that the rate of evaporation (loss of volume V) should be proportional to surface area A. Thus we are told that in a short time Δt the volume changes by ΔV, with

$$\Delta V = -cA\Delta t \tag{1}$$

Here c is some constant yet to be found, since we only know the rate is "proportional to surface area." Dividing through by Δt and taking the limit as Δt goes to zero, we derive the differential equation

$$\frac{dV}{dt} = -cA \tag{2}$$

Let us introduce R, the radius of the mothball. Using $V = 4\pi R^3/3$ and $A = 4\pi R^2$, we can put everything in terms of the radius R, giving

$$4\pi R^2 \frac{dR}{dt} = -4\pi R^2 c \tag{3}$$

or, since there is a nice cancellation of $4\pi R^2$ on both sides,

$$\frac{dR}{dt} = -c \tag{4}$$

That is, the radius R is just a linear function of time! This makes it easy to use the given information to complete the problem. We see, for example, that $c = 0.5$ cm/month, the constant rate of change of R.

(p. 556, #13) Water leaks from a vertical cylindrical tank through a small hole in its base at a rate proportional to the square root of the volume of water remaining. If the tank initially contains 200 liters, and 20 liters leak out during the first day, when will the tank be half empty? How much water will there be after 4 days?

Solution: We are told that in a short time Δt, the volume V changes by

$$\Delta V = c\sqrt{V}\Delta t \tag{5}$$

Here the constant c is to be found later from data, since we only know that the rate is "proportional to $\sqrt{V}$". Dividing through by Δt and letting Δt go to zero, we have the differential equation

$$\frac{dV}{dt} = -c\sqrt{V} \tag{6}$$

Solving by separation of variables we find

$$\int_{V_0}^{V} V^{-1/2} dV = -c \int_0^t dt \tag{7}$$

or

$$2\sqrt{V} = 2\sqrt{V_0} - ct \tag{8}$$

Solving for $V(t)$ we find

$$V = (\sqrt{V_0} - c't)^2 \tag{9}$$

where $c' = c/2$ is just another constant. From the given information we know $V_0 = 200$ liters, and $V(1) = 180 = (\sqrt{200} - c')^2$, so $c' = \sqrt{200} - \sqrt{180} \approx 0.726$. Knowing c' we can predict the future behavior. For example V is down to half its original value when $\sqrt{200} - c't = 10$, i.e., when $t = (\sqrt{200} - 10)/c' \approx 5.7$ days.