Math 324 Problem 10 Solution 4 March 2004

$$V = V_r\, dr + V_\theta\, r\, d\theta + V_\varphi\, r \sin\theta\, d\varphi$$

$$dV = \left[\frac{\partial}{\partial r}(V_\theta r) - \frac{\partial V_r}{\partial \theta}\right] dr\, d\theta + \left[r \frac{\partial (V_\varphi \sin\theta)}{\partial\theta} - r \frac{\partial V_\theta}{\partial \varphi}\right] d\theta\, d\varphi$$
$$+ \left[\frac{\partial V_r}{\partial \varphi} - \frac{\partial}{\partial r}(V_\varphi r) \sin\theta\right] d\varphi\, dr$$

$$-* dV = \left[\frac{\partial}{\partial r}(V_\theta r) - \frac{\partial V_r}{\partial \theta}\right] \sin\theta\, d\varphi + \left[r \frac{\partial (V_\varphi \sin\theta)}{\partial\theta} - r \frac{\partial V_\theta}{\partial \varphi}\right] \frac{1}{r^2 \sin\theta}\, dr$$
$$+ \left[\frac{\partial V_r}{\partial \varphi} - \frac{\partial}{\partial r}(V_\varphi r) \sin\theta\right] \frac{1}{\sin\theta}\, d\theta$$

$$\left[\begin{array}{l} \text{using } * dr\, d\theta = (dr\, d\theta,\ r^2 \sin\theta\, dr\, d\theta\, d\varphi) = -\sin\theta\, d\varphi \\ \quad * d\theta\, d\varphi = (d\theta\, d\varphi,\ r^2 \sin\theta\, dr\, d\theta\, d\varphi) = \dfrac{-1}{r^2 \sin\theta}\, dr \\ \quad * d\varphi\, dr = (d\varphi\, dr,\ r^2 \sin\theta\, dr\, d\theta\, d\varphi) = \dfrac{-1}{\sin\theta}\, d\theta \end{array}\right]$$

Now write this explicitly in terms of the orthonormal basis dr, $r\,d\theta$, $r\sin\theta\, d\varphi$:

$$-* dV = \frac{1}{r\sin\theta}\left[\frac{\partial (V_\varphi \sin\theta)}{\partial\theta} - \frac{\partial V_\theta}{\partial\varphi}\right] dr$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial V_r}{\partial\varphi} - \frac{\partial}{\partial r}(V_\varphi r)\right] r\, d\theta$$
$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(V_\theta r) - \frac{\partial V_r}{\partial\theta}\right] r \sin\theta\, d\varphi$$

The components of the curl with respect to this basis are the above coefficients.