Math 327 Problem 15 Solution 30 March 2004

Let the path C_0 be $x(u) = 0$, $y(u) = u$, $0 \le u \le 1$.

Then to find where it goes under $V = y\frac{\partial}{\partial x}$, we must solve

$$\left.\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= 0 \end{aligned}\right\} \text{ with } x_0 = 0,\ y_0 = u.$$

The solution is clearly $x_t = ut$, $y_t = u$.

Then
$$\int_{C_t} P(x,y)\,dx + Q(x,y)\,dy$$
$$= \int_0^1 \left[P(ut,u)\frac{dx_t}{du} + Q(ut,u)\frac{dy_t}{du} \right] du$$
$$= \int_0^1 \left[P(ut,u)\,t + Q(ut,u) \right] du$$

and hence
$$\frac{d}{dt}\int_{Ct} P(x,y)\,dx + Q(x,y)\,dy \Big|_{t=0} = \cancel{\int_0^1 \frac{\partial P}{\partial x}(ut,u)\,ut}$$
$$= \int_0^1 \left[\frac{\partial P}{\partial x}(ut,u)\cdot ut + P(ut,u) + \frac{\partial Q}{\partial x}(ut,u)\cdot u \right] du \Big|_{t=0}$$
$$= \int_0^1 \left[\cancel{\frac{\partial P}{\partial x}(0,u)} + P(0,u) + \frac{\partial Q}{\partial x}(0,u)\cdot u \right] du$$

Alternatively, $\left[\mathcal{L}_V (P\,dx + Q\,dy)\right]\left(\frac{\partial}{\partial y}\right) = VQ + P\,dx\left(\left[\frac{\partial}{\partial y}, V\right]\right) + Q\,dy\left(\left[\frac{\partial}{\partial y}, V\right]\right)$

$$= y\frac{\partial Q}{\partial x} + P$$

so $\displaystyle\int_{C_0} \mathcal{L}_V\left[P\,dx + Q\,dy\right] = \int_0^1 \left[P(0,y) + y\frac{\partial Q}{\partial x}(0,y) \right] dy$, agrees!