Math 324 Problem 16 Solution 1 April 2004

Let $V = V_1\,dx^1 + \text{c.p.}$

Here 'c.p.' means 'cyclic permutations', i.e., the terms generated from what is explicitly there already by the substitutions $1 \to 2 \to 3 \to 1$. Thus each term explicitly written implies 3 terms in all. Then we compute

$$dV = \frac{\partial V_1}{\partial x^2}\,dx^2 dx^1 + \frac{\partial V_1}{\partial x^3}\,dx^3 dx^1 + \text{c.p.}$$

$$*dV = \frac{\partial V_1}{\partial x^2}\,dx^3 - \frac{\partial V_1}{\partial x^3}\,dx^2 + \text{c.p.} = \left(\frac{\partial V_1}{\partial x^2} - \frac{\partial V_2}{\partial x^1}\right) dx^3 + \text{c.p.}$$

$$d{*}dV = \frac{\partial}{\partial x^1}\left(\frac{\partial V_1}{\partial x^2} - \frac{\partial V_2}{\partial x^1}\right) dx^1 dx^3 + \frac{\partial}{\partial x^2}\left(\frac{\partial V_1}{\partial x^2} - \frac{\partial V_2}{\partial x^1}\right) dx^2 dx^3 + \text{c.p.}$$

$$*d{*}dV = \frac{\partial}{\partial x^1}\left(\frac{\partial V_1}{\partial x^2} - \frac{\partial V_2}{\partial x^1}\right) dx^2 - \frac{\partial}{\partial x^2}\left(\frac{\partial V_1}{\partial x^2} - \frac{\partial V_2}{\partial x^1}\right) dx^1 + \text{c.p.}$$

$$= \left[\frac{\partial}{\partial x^3}\left(\frac{\partial V_3}{\partial x^1} - \frac{\partial V_1}{\partial x^3}\right) - \frac{\partial}{\partial x^2}\left(\frac{\partial V_1}{\partial x^2} - \frac{\partial V_2}{\partial x^1}\right)\right] dx^1 + \text{c.p.}$$

Also $*V = V_1\,dx^2 dx^3 + \text{c.p.}$

$$d{*}V = \frac{\partial V_1}{\partial x^1}\,dx^1 dx^2 dx^3 + \text{c.p.}$$

$$*d{*}V = -\frac{\partial V_1}{\partial x^1} + \text{c.p.}$$

Note equivalence under c.p.!

$$-\frac{\partial^2 V_3}{\partial x^1 \partial x^3}\,dx^1 \qquad -\frac{\partial^2 V_2}{\partial x^1 \partial x^2}\,dx^1$$

$$d{*}d{*}V = -\frac{\partial^2 V_1}{\partial (x^1)^2}\,dx^1 - \frac{\partial^2 V_1}{\partial x^2 \partial x^1}\,dx^2 - \frac{\partial^2 V_1}{\partial x^3 \partial x^1}\,dx^3 + \text{c.p.}$$

Thus

$$-*d{*}dV - d{*}d{*}V = \left[\frac{\partial^2 V_1}{\partial (x^1)^2} + \frac{\partial^2 V_1}{\partial (x^2)^2} + \frac{\partial^2 V_1}{\partial (x^3)^2} + \frac{\partial^2 V_3}{\partial x^3 \partial x^1} + \frac{\partial^2 V_2}{\partial x^2 \partial x^1} - \frac{\partial^2 V_3}{\partial x^1 \partial x^3} - \frac{\partial^2 V_2}{\partial x^1 \partial x^2}\right] dx^1 + \text{c.p.}$$

$$= (\nabla^2 V_1)\,dx^1 + \text{c.p.}$$