Math 324 Problem 18 Solution 8 Apr 2014

$$V = (x^2 - y^2)\,dx - 2xy\,dy$$

Check $\quad \operatorname{div} V = \frac{\partial}{\partial x}(x^2-y^2) + \frac{\partial}{\partial y}(-2xy) = 2x - 2x = 0$ ✓

Check $\quad \nabla^2 V_x = \nabla^2(x^2-y^2) = 2-2 = 0$

$\quad \nabla^2 V_y = \nabla^2(-2xy) = 0$

$\quad \nabla^2 V_z = 0$

That is $dP = 0$ so $P = \text{constant}$. ✓

All together this checks that V is a Stokes flow.

Now take V as a tangent vector, $(x^2-y^2)\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y}$

$$\tfrac{1}{2}\mathcal{L}_V g\left(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial x}\right) = g\left(\left[\tfrac{\partial}{\partial x}, V\right], \tfrac{\partial}{\partial x}\right) = g\left(2x\tfrac{\partial}{\partial x} - 2y\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial x}\right) = 2x$$

$$\begin{aligned}
\tfrac{1}{2}\mathcal{L}_V g\left(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}\right) &= \tfrac{1}{2} g\left(\left[\tfrac{\partial}{\partial x}, V\right], \tfrac{\partial}{\partial y}\right) + \tfrac{1}{2} g\left(\tfrac{\partial}{\partial x}, \left[\tfrac{\partial}{\partial y}, V\right]\right) \\
&= \tfrac{1}{2} g\left(2x\tfrac{\partial}{\partial x} - 2y\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial y}\right) + \tfrac{1}{2} g\left(\tfrac{\partial}{\partial x}, -2y\tfrac{\partial}{\partial x} - 2x\tfrac{\partial}{\partial y}\right) \\
&= -y - y = -2y
\end{aligned}$$

$$\tfrac{1}{2}\mathcal{L}_V g\left(\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial y}\right) = g\left(\left[\tfrac{\partial}{\partial y}, V\right], \tfrac{\partial}{\partial y}\right) = g\left(-2y\tfrac{\partial}{\partial x} - 2x\tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial y}\right) = -2x$$

Thus the rate of strain tensor is

$$e = \begin{pmatrix} 2x & -2y & 0 \\ -2y & -2x & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It clearly has zero trace: $\quad 2x - 2x = 0$.