Math 324 Problem 20 Solution 15 April 2014

Let $P = \dfrac{\cos\theta}{r^2}$

Then $u = dP = -\dfrac{\sin\theta}{r^2}\,d\theta - \dfrac{2\cos\theta}{r^3}\,dr$

As a tangent vector this is $-\dfrac{\sin\theta}{r^4}\dfrac{\partial}{\partial\theta} - \dfrac{2\cos\theta}{r^3}\dfrac{\partial}{\partial r}$

Recalling the metric tensor in spherical polar coordinates,

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$ we compute

$$e_{rr} = \tfrac{1}{2}\mathcal{L}_u g\left(\tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial r}\right) = g\left(\left[\tfrac{\partial}{\partial r}, u\right], \tfrac{\partial}{\partial r}\right) = \frac{6\cos\theta}{r^4}$$

$$e_{r\theta} = \tfrac{1}{2}\mathcal{L}_u g\left(\tfrac{\partial}{\partial r}, \tfrac{\partial}{\partial \theta}\right) = \tfrac{1}{2} g\left(\left[\tfrac{\partial}{\partial r}, u\right], \tfrac{\partial}{\partial \theta}\right) + \tfrac{1}{2} g\left(\tfrac{\partial}{\partial r}, \left[\tfrac{\partial}{\partial \theta}, u\right]\right)$$

$$= \tfrac{1}{2}\cdot\frac{4\sin\theta}{r^5}\cdot r^2 + \tfrac{1}{2}\,\frac{2\sin\theta}{r^3} = \frac{3\sin\theta}{r^3}$$

$$e_{\theta\theta} = \tfrac{1}{2}\mathcal{L}_u g\left(\tfrac{\partial}{\partial \theta}, \tfrac{\partial}{\partial \theta}\right) = \tfrac{1}{2} u r^2 + g\left(\left[\tfrac{\partial}{\partial \theta}, u\right], \tfrac{\partial}{\partial \theta}\right)$$

$$= -\frac{2\cos\theta}{r^2} - \frac{\cos\theta}{r^4}\cdot r^2 = -\frac{3\cos\theta}{r^2}$$

$$e_{r\varphi} = e_{\theta\varphi} = 0$$

$$e_{\varphi\varphi} = \tfrac{1}{2} u\, r^2 \sin^2\theta = -\frac{2\cos\theta}{r^2}\cdot\sin^2\theta - \frac{1}{2}\frac{\sin\theta}{r^4}\, r^2\, 2\sin\theta\cos\theta$$

$$= -\frac{3\cos\theta}{r^2}\sin^2\theta$$

so $$e_{ij} = \begin{pmatrix} \dfrac{6\cos\theta}{r^4} & \dfrac{3\sin\theta}{r^3} & 0 \\ \dfrac{3\sin\theta}{r^3} & -\dfrac{3\cos\theta}{r^2} & 0 \\ 0 & 0 & -\dfrac{3\cos\theta}{r^2}\sin^2\theta \end{pmatrix}$$

Check: the trace $g^{ij} e_{ij} = 0$.