Math 324 Problem 22 Solution 22 Apr 2004

$$0 = \nabla^2 \left(r^k P_k(\theta, \varphi) \right)$$ ($r^k P_k$ is a harmonic polynomial)

$$= \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} r^k \; P_k + r^k \frac{L^2}{r^2} P_k(\theta, \varphi)$$

(here L^2 is the part of ∇^2 that takes derivatives w.r.t. angles)

$$= k(k+1) r^{k-1} P_k + r^{k-1} L^2 P_k(\theta, \varphi)$$

Thus $L^2 P_k = -k(k+1) P_k$ is a property of spherical harmonics P_k.

Now try

$$\nabla^2 \left(\frac{P_k}{r^{k+1}} \right) = \nabla^2 \left(r^{-k-1} P_k \right)$$

$$= \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} r^{-k-1} P_k + r^{-k-1} \frac{1}{r^2} L^2 P_k$$

$$= (-k-1)(-k) r^{-k-3} P_k + r^{-k-3} \left(-k(k+1) P_k \right)$$

$$= 0$$

so $\frac{P_k}{r^{k+1}}$ is also harmonic.