Math 324 Problem 23 Solution 27 Apr 2004

Match $V = \cos\theta\, dr - R\sin\theta\, d\theta$ on $r = R$

(a) using the interior flows (from $\chi_1 = r\cos\theta$)

$$W_{1,3} = d\chi_1 = \cos\theta\, dr - r\sin\theta\, d\theta$$

and $$W_{1,1} = \frac{4}{2\cdot 2\cdot 5} r^2 (\cos\theta\, dr - r\sin\theta)\, d\theta - \frac{r}{2\cdot 5} r\cos\theta\, dr$$

$$= \frac{1}{10} r^2 \cos\theta\, dr - \frac{1}{5} r^3 \sin\theta\, d\theta$$

Take $A W_{1,1} + B W_{1,3}$ on $r = R$, and require that it match V:

$$\left(\frac{A}{10} R^2 + B\right)\cos\theta\, dr + \left(-\frac{A}{5} R^3 - BR\right)\sin\theta\, d\theta = \cos\theta\, dr - R\sin\theta\, d\theta$$

i.e. $\frac{A}{10} R^2 + B = 1$, $-\frac{A}{5} R^3 - BR = -R$. The solution is $A = 0$, $B = 1$.

(It is simpler to say this in Cartesian coordinates: the flow $W_{1,3}$, which <u>is</u> the interior flow, is just dz, translation in the z direction.)

(b) match exterior flow (using $\chi_{-2} = \frac{\cos\theta}{r^2}$)

$$W_{-2,3} = -\frac{2\cos\theta}{r^3} dr - \frac{\sin\theta}{r^2} d\theta$$

$$W_{-2,1} = \frac{1}{2(-1)(-1)}\left(-\frac{2\cos\theta}{r} dr - \sin\theta\, d\theta\right) - \frac{(-2)}{(-1)(-1)} \frac{\cos\theta}{r^2} r\, dr$$

$$= \frac{\cos\theta}{r} dr - \frac{1}{2}\sin\theta\, d\theta$$

Taking $A W_{-2,1} + B W_{-2,3}$, require

$$\left(\frac{A}{R} - \frac{2B}{R^3}\right)\cos\theta\, dr + \left(-\frac{A}{2} - \frac{B}{R^2}\right)\sin\theta\, d\theta = \cos\theta\, dr - R\sin\theta\, d\theta$$

The solution is $A = \frac{3}{2} R$, $B = \frac{1}{4} R^3$.