Maths 324 Problem 24 Solution 29 Apr 2004

(a) if $\chi_2 = \frac{1}{2}(-x^2-y^2+2z^2) = \frac{1}{2}A_{ij}x^ix^j$

$= \frac{r^2}{2}(-\sin^2\theta\cos^2\varphi - \sin^2\theta\sin^2\varphi + 2\cos^2\theta)$

$= r^2\left(\frac{2\cos^2\theta - \sin^2\theta}{2}\right) = r^2\left(\frac{3\cos^2\theta-1}{2}\right)$

Then $P_2 = \frac{3\cos^2\theta - 1}{2}$

(b) $V = d\chi_2 = -x\,dx - y\,dy + 2z\,dz$

Since $e_{ij} = \frac{1}{2}\left(\frac{\partial V_i}{\partial x^j} + \frac{\partial V_j}{\partial x^i}\right)$ in Cartesian coordinates,

and $V_i = \frac{\partial \chi_2}{\partial x^i}$, we have $e_{ij} = \frac{\partial^2 \chi_2}{\partial x^i \partial x^j} = A_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Then the rate of energy dissipation per unit volume is $2\eta e_{ij}e_{ij} = 2\eta(1^2+1^2+2^2) = 12\eta$ (constant)

and in a sphere of volume $\frac{4\pi}{3}\rho^3$, energy is dissipated at a rate $12\eta \cdot \frac{4\pi}{3}\rho^3 = 16\pi\eta\rho^3$.

(c) According to Einstein, the normal component of stress on the sphere of radius ρ does work at the rate $8\eta\rho^3 \int_0^{2\pi}\int_{-1}^{1} P_2^2 \, d(\cos\theta)\, d\varphi = 16\pi\eta\rho^3 \int_{-1}^{1} P_2^2 \, d(\cos\theta)$

But $\int_{-1}^{1} P_2^2 \, d(\cos\theta) = \int_{-1}^{1} (3x^2-1)^2 dx = \frac{2}{5}$

so more energy is dissipated than is coming in.
(The explanation is that more energy is coming in than this: the tangential components of stress also do work.)