Math 324 Problem 3 Solution 9 Feb 2004

$x = r \sin\theta \cos\varphi$

$y = r \sin\theta \sin\varphi$

$z = r \cos\theta$

(a) Using $g\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\right) = g\left(\frac{\partial x^k}{\partial q^i}\frac{\partial}{\partial x^k}, \frac{\partial x^\ell}{\partial q^j}\frac{\partial}{\partial x^\ell}\right) = \frac{\partial x^k}{\partial q^i}\frac{\partial x^\ell}{\partial q^j}\delta_{k\ell}$

with $(q^1, q^2, q^3) = (r, \theta, \varphi)$ we find

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 = \sin^2\theta(\cos^2\varphi + \sin^2\varphi) + \cos^2\theta = 1$$

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right) = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \theta} = r\sin\theta\cos\theta + \cos\theta(-r\sin\theta) = 0$$

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \varphi}\right) = \frac{\partial x}{\partial r}\frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \varphi} + \frac{\partial z}{\partial r}\frac{\partial z}{\partial \varphi} = r\sin^2\theta\left(\cos\varphi(-\sin\varphi) + \sin\varphi\cos\varphi\right) = 0$$

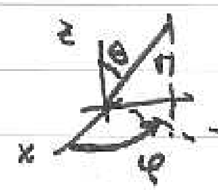
$$g\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\right) = \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 = r^2\cos^2\theta(\cos^2\varphi + \sin^2\varphi) + r^2\sin^2\theta = r^2$$

$$g\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}\right) = \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \varphi} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \varphi} = r^2\cos\theta\sin\theta\cos\varphi(-\sin\varphi) + r^2\cos\theta\sin\theta\cos\varphi\sin\varphi = 0$$

$$g\left(\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \varphi}\right) = \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 = r^2\sin^2\theta(\sin^2\varphi + \cos^2\varphi) = r^2\sin^2\theta$$

i.e. $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$ as a matrix of components.

(b) The vector field $\frac{\partial}{\partial \varphi}$ has a flow which is rigid rotation about the $z$-axis




The speed of this rotation is just the distance from the $z$-axis, which is $r\sin\theta$,

i.e. $\sqrt{g\left(\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \varphi}\right)}$.