Math 324 Problem 5 Solution 17 Feb 2004

Compute $\mathcal{L}_v g(X,Y) = v_g(X,Y) + g([X,v],Y) + g(X,[Y,v])$ for $v = \frac{\partial}{\partial r}$, X and Y all pairs from $\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}\right)$.

Note $[v,X] = [v,Y] = 0$ in all cases, since it is always one of $\left[\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right], \left[\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right], \left[\frac{\partial}{\partial r}, \frac{\partial}{\partial \varphi}\right]$, and $\{r, \theta, \varphi\}$ are the coordinates. Thus we have only $\frac{\partial}{\partial r} g(X,Y)$. Since

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

the only non-zero components are

$$\mathcal{L}_{\frac{\partial}{\partial r}} g\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\right) = 2r$$

and $\mathcal{L}_{\frac{\partial}{\partial r}} g\left(\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \varphi}\right) = 2r\sin^2\theta$

i.e. $$\mathcal{L}_{\frac{\partial}{\partial r}} g = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2r & 0 \\ 0 & 0 & 2r\sin^2\theta \end{pmatrix}$$