Math 324 Problem & Solution 26 Feb 2004

1a) $x = r\cos\theta$, $y = r\sin\theta$ $\Rightarrow$

$$dx = \cos\theta\, dr - r\sin\theta\, d\theta$$
$$dy = \sin\theta\, dr + r\cos\theta\, d\theta$$

Then
$$dx\,dy = (\cos\theta\, dr - r\sin\theta\, d\theta)(\sin\theta\, dr + r\cos\theta\, d\theta)$$
$$= r\cos^2\theta\, dr\,d\theta - r\sin^2\theta\, d\theta\,dr$$
$$= r(\cos^2\theta + \sin^2\theta)\, dr\,d\theta = r\,dr\,d\theta$$

(b) $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$ $\Rightarrow$

$$dx = \sin\theta\cos\varphi\, dr + r\cos\theta\cos\varphi\, d\theta - r\sin\theta\sin\varphi\, d\varphi$$
$$dy = \sin\theta\sin\varphi\, dr + r\cos\theta\sin\varphi\, d\theta + r\sin\theta\cos\varphi\, d\varphi$$
$$dz = \cos\theta\, dr - r\sin\theta\, d\theta$$

Then
$$dx\,dy = r\sin\theta\cos\theta\sin\varphi\cos\varphi\,(dr\,d\theta + d\theta\,dr)$$
$$+ r\sin^2\theta\cos^2\varphi\, dr\,d\varphi - r\sin^2\theta\sin^2\varphi\, d\varphi\,dr$$
$$+ r^2\sin\theta\cos\theta\cos^2\varphi\, d\theta\,d\varphi - r^2\sin\theta\sin^2\varphi\cos\theta\, d\varphi\,d\theta$$
$$= r\sin^2\theta\, dr\,d\varphi + r^2\sin\theta\cos\theta\, d\theta\,d\varphi$$

and
$$dx\,dy\,dz = (r\sin^2\theta)(-r\sin\theta)\, dr\,d\varphi\,d\theta + r^2\sin\theta\cos^2\theta\, d\theta\,d\varphi\,dr$$
$$= r^2\sin\theta(\sin^2\theta + \cos^2\theta)\, dr\,d\theta\,d\varphi$$
$$= r^2\sin\theta\, dr\,d\theta\,d\varphi$$

using antisymmetry of multiplication: $dr\,d\varphi\,d\theta = -dr\,d\theta\,d\varphi$, etc.