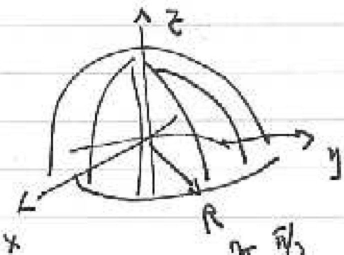
Math 324 Problem 9 Solution 2 March 2004

(a)



$$\iint_{\text{hemisphere}} \hat{z}\cdot\hat{n}\, dA$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{zR^2}{R}\sin\theta\, d\theta\, d\varphi = R^2\int_0^{2\pi}\int_0^{\pi/2} \cos\theta\sin\theta\, d\theta\, d\varphi$$

[using $\hat{n} = \hat{r} = \left(\frac{x}{R}, \frac{y}{R}, \frac{z}{R}\right)$ and $z = R\cos\theta$]

$$= 2\pi R^2 \int_0^{\pi/2} \cos\theta\sin\theta\, d\theta$$

$$= 2\pi R^2\, \frac{1}{2}\sin^2\theta \Big|_0^{\pi/2} = \pi R^2$$

(b) This looks suspiciously like a computation we did of $\int_{\text{hemisphere}} dx\, dy$. In fact, if we evaluate on tangent vectors to the sphere (so that we can omit dr),

$$dx = R\cos\theta\cos\varphi\, d\theta - R\sin\theta\sin\varphi\, d\varphi$$

$$dy = R\cos\theta\sin\varphi\, d\theta + R\sin\theta\cos\varphi\, d\varphi$$

so $dx\, dy = R^2\cos\theta\sin\theta(\cos^2\varphi + \sin^2\varphi)\, d\theta\, d\varphi = R^2\cos\theta\sin\theta\, d\theta\, d\varphi$

If we think about it geometrically, we see this is no accident. dA assigns area to a little piece of surface, but $\hat{n}\cdot\hat{z} = \cos\theta$ projects this area, giving just the area one would see, foreshortened, looking down the z-axis. On the other hand, $dx\,dy$ assigns the area projected into the x-y plane. These are the same!