THREE METHODS OF CONSTRUCTING RATIONAL REPRESENTATIONS OF KNOTS

CRAIG PHILLIPS

Abstract. This paper will discuss different ways of finding rational parameterizations of knots in $\mathbb{R}^3$. We will first show a way of constructing torus knots, and then two ways of converting polynomial representations of knots into rational representations, the advantage of which is the possibility of one-point compactification in $\mathbb{R}^3$. This work was completed as part of the Mount Holyoke Summer Mathematics Institute, an NSF funded REU Program, under the advisement of Alan Durfee and Donal O'Shea. It was also completed while working with a research group consisting of David Clark, Donovan McFeron, Virginia Peterson, and Alexandra Zuser. Additional funding was also provided by the Mt Holyoke College Department of Mathematics.

1. Introduction

We know that any knot can be parameterized by a polynomial in $\mathbb{R}^3$[4, page 13]. Any such knots can be compactified $\mathbb{R}^3 \cup \{\infty\}$. Rational polynomial knots have the advantage of being able to be compactified in $\mathbb{R}^3$ itself, without adding the point at ∞.

2. A Rational Parameterization of a Torus

First, we define $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$

2.1. **Proposition.** *The function* $\Phi(r,s) : \mathbb{R}_\infty \times \mathbb{R}_\infty \to \mathbb{R}^3$, *defined as* $\Phi(r,s) =$

$$\left(\left(\frac{1-r^2}{1+r^2}\right)\left(3+\frac{1-s^2}{1+s^2}\right), \left(\frac{2r}{1+r^2}\right)\left(3+\frac{1-s^2}{1+s^2}\right), \left(\frac{2s}{1+s^2}\right)\right)$$

is a parameterization of a torus in $\mathbb{R}^3$.

Proof. We will construct the torus from a rational parameterization of the circle in $\mathbb{R}^2$. First, we recall the following rational parameterization $f : \mathbb{R}_\infty \to S^1$ of a circle in $\mathbb{R}^2$, where $f(t) = (x(t), y(t))$, and:

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$$\begin{aligned} x(t) &= \frac{1-t^2}{1+t^2} \\ y(t) &= \frac{2t}{1+t^2} \end{aligned}$$

It is important to note that f is a bijection.

Now, we can construct a torus in $\mathbb{R}^3$ from x and y. We begin with a circle of radius 3 in $\mathbb{R}^3$ in the xy-plane centered at the origin. It is parameterized by

$$\Theta(r) = \left(3\left(\frac{1-r^2}{1+r^2}\right), 3\left(\frac{2r}{1+r^2}\right), 0\right)$$

To generate a torus, we construct a tube of radius 1 about the circle. We thus need to add to Θ another function, $\Psi(r,s)$, which is, for each r, a unit circle about the origin in the plane normal to $\Theta(r)$.

We will use as an orthonormal basis for that plane:

$$(0,0,1), \left(\frac{1-r^2}{1+r^2}, \frac{2r}{1+r^2}, 0\right).$$

Thus, we can now define $\Psi(r,s) =$

$$\left(\left(\frac{1-r^2}{1+r^2}\right)\left(\frac{1-s^2}{1+s^2}\right), \left(\frac{2r}{1+r^2}\right)\left(\frac{1-s^2}{1+s^2}\right), \left(\frac{2s}{1+s^2}\right)\right)$$

Our final torus, $\Phi = \Theta + \Psi$ is then:

$$\Phi(r,s) =$$

$$\left(\left(\frac{1-r^2}{1+r^2}\right)\left(3+\frac{1-s^2}{1+s^2}\right), \left(\frac{2r}{1+r^2}\right)\left(3+\frac{1-s^2}{1+s^2}\right), \left(\frac{2s}{1+s^2}\right)\right)$$

□

3. KNOTTING ON THE TORUS

In order to find a knot on the torus parameterized above, we will restrict r and s to functions of the same variable, t.

We can recognize, intuitively, that as r goes from $-\infty$ to $+\infty$ where $s = s_0$ is fixed, a single longitude is generated. Similarly, as s goes from $-\infty$ to $+\infty$ where $r = r_0$ is fixed, a single meridian is generated.

Thus, in order to construct a (p,q)-torus knot, we will need to replace r and s by piecewise continuous functions $r(t)$ and $s(t)$ which go from $-\infty$ to $+\infty$ q and p times, respectively. More formally, we state the following:

3.1. **Proposition.** *$\Psi(r(t), s(t))$ represents a (p,q)-torus knot if the functions $r(t)$ and $s(t)$, where $r : \mathbb{R}_\infty \to \mathbb{R}_\infty$ and $s : \mathbb{R}_\infty \to \mathbb{R}_\infty$, satisfy the following conditions:*

1. $r(t)$ (resp., $s(t)$) is continuous except at exactly q (resp., p) points, $m_1, m_2, ..., m_q$ (resp., $n_1, n_2, ..., n_p$) where $r(m_i) = \infty, 1 \leq i \leq q$ $[s(n_i) = \infty, 1 \leq i \leq p]$.
2. Either $\lim_{t \to m_i^+} r(t) = -\infty$ and $\lim_{t \to m_i^-} r(t) = +\infty$ for all $1 \leq i \leq q$ or $\lim_{t \to m_i^+} r(t) = +\infty$ and $\lim_{t \to m_i^-} r(t) = -\infty$ for all $1 \leq i \leq q$. The analogous condition holds for $s(t)$.[1]
3. $\lim_{t \to \infty} r(t)$ exists and is equal to $r(\infty)$. Similarly, $\lim_{t \to \infty} s(t)$ exists and is equal to $s(\infty)$.
4. There do not exist distinct $t_1, t_2 \in \mathbb{R}_\infty$ such that $r(t_1) = r(t_2)$ and $s(t_1) = s(t_2)$.

Conditions (1) and (2) assure that $\Phi(r(t), s(t))$ goes around the meridian and the longitude of the torus p and q times, respectively. Condition (3) assures that $\Phi(r(t), s(t))$ closes to the same point at ∞, and (4) assures that there are no intersection points on $\Phi(r(t), s(t))$.

3.2. **Example.**

(3,2)-torus knot. Consider the following equations:

$$r(t) = \frac{1}{t-3} + \frac{1}{t+3}$$

$$s(t) = \frac{1}{t-4} + \frac{1}{t} + \frac{1}{t+4}$$

It can be easily checked (using Maple for condition 4) that r and s satisfy the 5 conditions.

Thus, $\Phi(r(t), s(t))$ is a rational parameterization of the $(3,2)$-torus knot. ☐

We have seen in this example r and s satisfying the above conditions, where $(p,q) = (3,2)$. However, an algorithmic method of constructing r and s that satisfy the above conditions for any p and q remains unknown.

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[1] Since we are working in $\mathbb{R}_\infty$, it is true that $+\infty = -\infty$. When saying that a limit is $+\infty$, we mean that the function increases without bounds, and when saying that a limit is $-\infty$, we mean that the function decreases without bounds.

4. INVERSION

In order to convert a polynomial representation of a knot into a rational polynomial representation of that same knot, consider the function $f : \mathbb{R}^3 - \{0\} \rightarrow \mathbb{R}^3 - \{0\}$ defined as follows:

$$f(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z)) = \\ \left(\frac{x}{x^2+y^2+z^2}, \frac{y}{x^2+y^2+z^2}, \frac{z}{x^2+y^2+z^2}\right)$$

One can see that f is a homeomorphism, and in fact a diffeomorphism. However, f is orientation-reversing, since its jacobian, Df,

$$\frac{1}{(x^2+y^2+z^2)^2}\begin{pmatrix} y^2+z^2-x^2 & -2y & -2z \\ -2x & x^2+z^2-y^2 & -2z \\ -2x & -2y & x^2+y^2-z^2 \end{pmatrix},$$

has determinant

$$\frac{-1}{(x^2+y^2+z^2)^3},$$

which is negative for all $(x,y,z) \in \mathbb{R}^3 - \{0\}$. Thus, the orientation of $\mathbb{R}^3 - \{0\}$ is reversed under f. We know then that the image $f(K)$ of any knot $K \in \mathbb{R}^3 - \{0\}$ is equivalent to K^m, or the *mirror image* of K. In order to preserve orientation, we replace the function f by a new function, $\Psi : (x,y,z) \mapsto (f_1(x,y,z), f_3(x,y,z), f_2(x,y,z))$. Clearly $\Psi : \mathbb{R}^3 - \{0\} \rightarrow \mathbb{R}^3 - \{0\}$ is also a homeomorphism. Since $D\Psi$ is obtained by switching two rows of Df, $|D\Psi|$ is positive for all $(x,y,z) \in \mathbb{R}^3$, so $\Psi : \mathbb{R}^3 - \{0\} \rightarrow \mathbb{R}^3 - \{0\}$ is an orientation-preserving homeomorphism; and any knot, $K \in \mathbb{R}^3 - \{0\}$, is equivalent to its image, $\Psi(K)$, under Ψ.

4.1. **Example.**

Shastri's Trefoil. Consider Shastri's parameterization of a trefoil [4, page 14], $g(t) = (x(t), y(t), z(t))$ where:

$$\begin{aligned} x(t) &= t^3 - 3t \\ y(t) &= t^4 - 4t^2 \\ z(t) &= t^5 - 10t \end{aligned}$$

Notice that when $t = 0$, $g(t) = (0, 0, 0)$, at which Ψ is undefined. In order to avoid this problem, we define $x_0(t) = x(t) + 8$, so that the $g_0(t) = (x_0(t), y(t), z(t)) \neq 0$ for all $t \in \mathbb{R}$.

Now, we can consider the parameterization $\Psi \circ g_0 : \mathbb{R} \to \mathbb{R}^3$. From what we have shown above, we know that the knot represented by $\Psi \circ g_0(\mathbb{R})$ is equivalent to the knot represented by $g_0(\mathbb{R})$. □

Another issue that we will discuss about the image of a polynomial knot under Ψ is the question of smoothness. It is clear, since Ψ is a diffeomorphism, that if a polynomial knot, $f(\mathbb{R})$, is smooth, then $\Psi \circ f(\mathbb{R})$ is also smooth. However, since f is unbounded in $\mathbb{R}^3$ (i.e. $\lim_{t\to+\infty} |\Psi \circ f(t)| = +\infty$ and $\lim_{t\to-\infty} |\Psi \circ f(t)| = +\infty$), we know that

$$\lim_{t\to+\infty} \Psi \circ f(t) = \lim_{t\to-\infty} \Psi \circ f(t) = (0, 0, 0).$$

Thus, by adding the point $(0, 0, 0)$ to $\Psi \circ f(\mathbb{R})$, we obtain a compact representation of the knot represented by $f(\mathbb{R})$. However, we must check the "smoothness" of $\Psi(t)$ at $(0, 0, 0)$. In other words, we want see if, or when, the direction of the derivative,

$$\frac{d(\Psi \circ f)/dt}{|d(\Psi \circ f)/dt|},$$

is the same as t approaches $+\infty$ and as t approaches $-\infty$.

4.2. **Proposition.** *Given a polynomial parameterization of a knot,* $f : \mathbb{R} \to \mathbb{R}^3 - \{0\}$, *the composition* $\Psi \circ f$ *is such that*

$$\lim_{t\to+\infty} \frac{d(\Psi \circ f)/dt}{|d(\Psi \circ f)/dt|} = \lim_{t\to-\infty} \frac{d(\Psi \circ f)/dt}{|d(\Psi \circ f)/dt|}$$

if and only if deg(f) *is odd.*

Proof. Let $f(t) = (x_1(t), x_2(t), x_3(t))$, where

$$x_1(t) = a_1 t^{n_1} + ...$$

$$x_2(t) = a_2 t^{n_2} + ...$$

$$x_3(t) = a_3 t^{n_3} + ...$$

where each "..." represents terms of a lesser degree in each polynomial, and a_1, a_2, and a_3 are all non-zero. (Thus, n_1, n_2, and n_3 are the degrees of x_1, x_2, and x_3, respectively.) Furthermore, we can assume without loss of generality that $n_1 \geq n_2 \geq n_3$. We will now consider three possible cases.

(i) $n_1 > n_2$

$\Psi \circ f = (y_1(t), y_2(t), y_3(t))$ can be defined as

$$y_1(t) = \frac{a_1 t^{n_1} + ...}{a_1^2 t^{2n_1} + ...}$$
$$y_2(t) = \frac{a_3 t^{n_3} + ...}{a_1^2 t^{2n_1} + ...}$$
$$y_3(t) = \frac{a_2 t^{n_2} + ...}{a_1^2 t^{2n_1} + ...}.$$

(It should be noted that the reason for this notation is that we will soon be taking the limit as $t \to \infty$, for which we will only be concerned with the highest term.)

We now need to differentiate $\Psi \circ f$. After much computation, we obtain:

$$y_1'(t) = \frac{-n_1 a_1^3 t^{-n_1-1} + ...}{a_1^4 + ...}$$
$$y_2'(t) = \frac{(n_2 a_1^2 a_2 - 2n_1 a_1^2 a_2) t^{-2n_1+n_2-1} + ...}{a_1^4 + ...}$$
$$y_3'(t) = \frac{(n_3 a_1^2 a_3 - 2n_1 a_1^2 a_3) t^{-2n_1+n_3-1} + ...}{a_1^4 + ...}$$

Finally, since all three of these components approach 0 as t approaches $+\infty$ and $-\infty$, we need to find the unit tangent vector, i.e. $\frac{d(\Psi \circ f)/dt}{|d(\Psi \circ f)/dt|}$. First, we find $|y'(t)|$:

$$|y'(t)| = \sqrt{\frac{n_1^2 a_1^6 t^{-2n_1-2} + ...}{a_1^8 + ...}}$$

Thus, we have:

$$\lim_{t \to +\infty} \frac{y_1'(t)}{|y'(t)|} = \lim_{t \to +\infty} \left(\frac{-n_1 a_1^3 t^{-n_1-1} + ...}{a_1^4 + ...} \right) \left(\sqrt{\frac{a_1^8 + ...}{n_1^2 a_1^6 t^{-2n_1-2} + ...}} \right)$$
$$= \left(\frac{-a_3}{|a_3|} \right) \left(\frac{t^{-n_1-1}}{|t^{-n_1-1}|} \right) = \begin{cases} -1 & \text{if } a_1 > 0 \\ 1 & \text{if } a_1 < 0 \end{cases},$$

$$\lim_{t \to +\infty} \frac{y_2'(t)}{|y'(t)|} =$$
$$\lim_{t \to +\infty} \left(\frac{(n_2 a_1^2 a_2 - 2n_1 a_1^2 a_2) t^{-2n_1+n_2-1} + ...}{a_1^4 + ...} \right) \left(\sqrt{\frac{a_1^8 + ...}{n_1^2 a_1^6 t^{-2n_1-2} + ...}} \right) = 0,$$

$$\lim_{t\to+\infty} \frac{y_3'(t)}{|y'(t)|} =$$

$$\lim_{t\to+\infty} \left(\frac{(n_3 a_1^2 a_3 - 2n_1 a_1^2 a_3)t^{-2n_1+n_3-1} + ...}{a_1^4 + ...} \right) \left(\sqrt{\frac{a_1^8 + ...}{n_1^2 a_1^6 t^{-2n_1-2} + ...}} \right) = 0.$$

It can be further observed that taking the limit as $t \to -\infty$ of the above expressions, all but the first will yield identical solutions, regardless of the degrees of the polynomials. However, if y_1 is of even degree, then $\frac{t^{-n_1-1}}{|t^{-n_1-1}|}$ goes to 1 as $t \to +\infty$ and to -1 as $t \to -\infty$. Thus, the theorem is proved.

The proofs for the other two possibilities ($n_1 = n_2 > n_3$) and ($n_1 = n_2 = n_3$) are similarly straight-forward and more computational, and will not be included here. ☐

5. Change of Coordinates in Projective Space (✠)

We will now, in a different way, construct from a polynomial representation of a knot, $K \in \mathbb{R}^3$, a bounded rational representation that can be compactified in $\mathbb{R}^3$. In order convert a polynomial parameterization of a knot, $f : t \mapsto (x(t), y(t), z(t))$, into a closed bounded rational representation of the same knot, we will use the embedding $h : \mathbb{R}^3_{x+} \to \mathbb{R}^3$, where $h : (x, y, z) \mapsto \left(\frac{z}{x}, \frac{y}{x}, \frac{1}{x}\right)$, and $\mathbb{R}^3_{x+} = \{(x, y, z) \in \mathbb{R}^3 | x > 0\}$. It can be easily checked that $|Dh| = \frac{1}{x^4}$, and is therefore always positive, showing that h is orientation-preserving. Of course, we will first have to modify f in order to assure that $f(\mathbb{R})$ is above the yz coordinate plane $\mathbb{R}$.

5.1. **Proposition.** *Any polynomial knot parameterization can be modified algorithmically in order to produce a similar polynomial that lies completely in* $\mathbb{R}^3_{x+}$.

Proof. First, we will assume without loss of generality that $deg(x) = deg(f)$, and let $m = deg(x)$. If $deg(x) < deg(f)$, we can just change coordinates so that $deg(x) = deg(f)$. If $deg(f)$ is odd, we will need to construct a new parameterization, $f_0 = (x_0(t), y(t), z(t))$, where $x_0(t) = x(t) + \varepsilon t^{m+1}$ for some sufficiently small $\varepsilon > 0$, which does not change any crossings in the x direction.

Since $x_0(t)$ is a polynomial of even degree, we now have that

$$\lim_{t\to+\infty} x_0(t) = \lim_{t\to-\infty} x_0(t) = +\infty.$$

Thus, there is at least one t_0 where $x_0(t_0) \leq x_0(t)$ for all $t \in \mathbb{R}$. We then add a sufficiently large constant C to $x_0(t)$ such that $x_1(t_0) = x_0(t_0) + C > 0$. This completes the proof. $\square$

Finally, we require that $deg(x) > deg(y)$ and $deg(x) > deg(z)$, so that $\lim_{t\to+\infty} f(t) = \lim_{t\to-\infty} f(t) = (0,0,0)$. If this is not the case, then we can add $cx(t)$ to $y(t)$, where $cx(t) + y(t)$ has degree less than $deg(x)$. In other words, we make c a constant so that the highest term of $cx(t)$ cancels out the highest term of $y(t)$. We do the same thing for $z(t)$. We can see that this transformation, when done to all of $\mathbb{R}^3_{x^+}$, is a homeomorphism, so we have not changed the knot type.

Now we can extend $f_1 : \mathbb{R} \to \mathbb{R}^3$ to the $f_1 : \mathbb{R}_\infty \to \mathbb{R}^3_{x+} \cup \infty$, where $f_1(\infty) = \infty$. Then, we extend h to $h : \mathbb{R}^3_{x+} \cup \infty \to \mathbb{R}^3$, where $h(\infty) = (0,0,0)$. Then, $h \circ f_1(\mathbb{R}_\infty) \in \mathbb{R}^3$ is a compact representation of the original knot, $f(K)$, and is completely contained in $\mathbb{R}^3$.

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RUTGERS THE STATE UNIVERSITY OF NEW JERSEY, NEW BRUNSWICK, NJ, 08903
E-mail address: `craigz@eden.rutgers.edu`