

# Transforming Trigonometric Knot Parameterizations into Rational Knot Parameterizations

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## Abstract

This paper develops a method for constructing rational parameterizations of knots, based on a trigonometric parameterization. It also introduces the class of torus knots and describes a method for constructing trigonometric and rational parameterizations of these knots.

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## 1 Introduction

In 1992, Shastri created polynomial parameterizations of several knots, and showed that every knot type has a representation which has a polynomial parameterization [5]. In 1995, Trautwein gave trigonometric parameterizations (called *Harmonic Knots* in his thesis) of all knots up through 7 crossings [7], and showed that they exist for all knots.<sup>2</sup>

Work on creating knot parameterizations has focused almost exclusively on polynomial and trigonometric parameterizations. At the Mt. Holyoke REU, we have focused on parameterizing knots with rational functions. We have taken many approaches, and this paper deals with one method of constructing rational parameterizations from trigonometric parameterizations. We will also show how to explicitly construct trigonometric, and therefore rational, parameterizations of the torus knots.

Throughout this paper (and in the following definition), a *smooth* map is one which is  $C^1$ .

**Definition** A *knot* is a smooth embedding of a circle in  $\mathbb{R}^3$ .

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<sup>2</sup>More information about harmonic knots can be found in Trautwein's thesis [6]. Theorem 1.29 specifically proves the existence of harmonic knots for all knot types.

In the literature, more general types of knots are sometimes allowed. Any knot in our sense is considered a tame knot. (See [2] for a further explanation of tame and wild knots.)

We are particularly interested in parametric representations of knots. There are two general forms of rational parameterizations which interest us. It is useful to define each form in terms of its domain. Trigonometric parameterizations use a finite interval, for which we define a smooth map  $f : (a, b) \rightarrow \mathbb{R}^3$  which is continuous on  $[a, b]$  and such that  $f(a) = f(b)$  and  $f'(a) = f'(b)$ . Here, the derivative denotes the appropriate one-sided limit. Rational parameterizations, as we will use them, require a smooth map  $f : \mathbb{R} \rightarrow \mathbb{R}^3$ . In this case, the one-sided limits of the derivatives of  $f$  must be equal as the function approaches  $\pm\infty$ . While these parameterizations use different intervals, it is possible for their images to be the same.

We can now begin constructing rational knot parameterizations.

## 2 Equating Trigonometric and Rational functions

Our goal is to find rational parameterizations of knots. Since a trigonometric parameterization exists for every knot, trigonometric parameterizations provide a useful starting point for constructing other parameterizations. If we can transform trigonometric functions into rational functions, then we are assured that a rational parameterization exists for every knot.

A knot is an embedding of a circle in  $\mathbb{R}^3$ , so we will consider two different parameterizations of a unit circle: one trigonometric, the other rational. Trigonometrically, we can set:

$$\begin{aligned}x(\theta) &= \cos \theta \\y(\theta) &= \sin \theta\end{aligned}$$

for  $-\pi < \theta < \pi$ .

Rationally, we can set:

$$\begin{aligned}x(t) &= \frac{1 - t^2}{1 + t^2} \\y(t) &= \frac{2t}{1 + t^2}\end{aligned}$$

Here we allow  $t$  to range over  $\mathbb{R}$ .

**Remark** This can be considered the “simplest” rational parameterization of the circle, in terms of the degrees of each function’s constituent polynomials, as well as its domain. It is possible to construct many more complex parameterizations which use finite intervals to draw complete circles. We could substitute many of these alternate parameterizations for the given rational parameterization and have only relatively minor algebraic changes throughout the rest of this paper. However, for our purposes, this relatively simple parameterization functions very well.

We can transform the trigonometric parameterization directly into the rational parameterization by using an appropriate change of parameters.

**Lemma 2.1** *Let  $\beta : (-\pi, \pi) \rightarrow \mathbb{R}^2$  be defined by*

$$\beta(t) = (\cos \theta, \sin \theta).$$

*Then the change of parameters  $\theta = 2 \arctan t$  transforms  $\beta$  into  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$  defined by*

$$\gamma(t) = \left( \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right).$$

**Proof** The change of parameters  $\theta = 2 \arctan t$  is differentiable and invertible on the entire real line, and its inverse  $t = \tan \frac{\theta}{2}$  exists and is also differentiable on the interval  $(-\pi, \pi)$ .

To show that this change of parameters does transform  $\beta$  into  $\gamma$ , we will work with each component separately, beginning with  $\cos \theta$ :

$$\begin{aligned} \cos \theta &= \cos 2 \cdot \frac{\theta}{2} = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} = \frac{1 - \tan^2 \frac{\theta}{2}}{\sec^2 \frac{\theta}{2}} \\ &= \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} \end{aligned}$$

Substituting  $\theta = 2 \arctan t$ ,

$$= \frac{1 - t^2}{1 + t^2}$$

Now, working with  $\sin \theta$ :

$$\begin{aligned} \sin \theta &= \sin 2 \cdot \frac{\theta}{2} = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = 2 \tan \frac{\theta}{2} \cos^2 \frac{\theta}{2} = \frac{2 \tan \frac{\theta}{2}}{\sec^2 \frac{\theta}{2}} \\ &= \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} \end{aligned}$$

Substituting  $\theta = 2 \arctan t$ ,

$$= \frac{2t}{1 + t^2}$$

Therefore,  $\theta = 2 \arctan t$  is a change of parameters which takes  $\beta$  to  $\gamma$ .  $\square$

**Remark** On the open interval  $(-\pi, \pi)$ , the trigonometric parameterization  $\beta$  misses exactly one point,  $(-1, 0)$ , on the circle. Similarly, when  $t$  ranges over  $\mathbb{R}$ , the rational parameterization  $\gamma$  misses exactly one point,  $(-1, 0)$ , on the circle. It is possible to patch  $\beta$  by extending its domain to include  $-\pi$  and  $\pi$ , noting that  $\beta(-\pi) = \beta(\pi)$ . It is also possible to patch  $\gamma$  if we extend its domain by adding a “point at infinity,” effectively making the domain of  $\gamma$  into the Real Projective Line, or  $\mathbb{R}P^1$ . We refer to this domain as “ $\mathbb{R} \cup \{\infty\}$ ” throughout this paper. This gives us a complete unit circle from both  $\beta$  and  $\gamma$ .

Lemma 2.1 allows us to replace  $\cos \theta$  with  $\frac{1-t^2}{1+t^2}$  and  $\sin \theta$  with  $\frac{2t}{1+t^2}$  when  $\theta$  is in the interval  $[-\pi, \pi]$ . If we let  $t$  range over  $\mathbb{R} \cup \{\infty\}$ , there will be no change in the resulting curve.

One problem with this transformation is that we can only transform  $\cos \theta$  and  $\sin \theta$ . We do not have an explicit way to transform  $\cos n\theta$  and  $\sin n\theta$ , which appear in many trigonometric parameterizations. To solve this problem, we need to find a way to reduce  $\cos n\theta$  and  $\sin n\theta$  to a polynomial in  $\cos \theta$  and  $\sin \theta$ .

**Lemma 2.2** *The functions  $\cos n\theta$  and  $\sin n\theta$  are polynomials in  $\cos \theta$  and  $\sin \theta$ .*

**Proof** Consider the rotation matrix

$$R_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

which will rotate a given point through an angle of  $\theta$  radians. We can represent a matrix which rotates a given point through  $n\theta$  radians in two ways.

First, we can simply use  $R_{n\theta}$ :

$$R_{n\theta} = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$$

Alternately, we can let  $M$  be the matrix product of  $R_\theta$  with itself  $n$  times, which rotates the point through  $\theta$  radians  $n$  times:

$$M = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \cdot \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \cdots \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

By its construction, each element of  $M$  must be a polynomial in  $\sin \theta$  and  $\cos \theta$ . Since  $R_{n\theta}$  and  $M$  are equal, the  $(i, j)$  entry of  $R_{n\theta}$  and the  $(i, j)$  entry of  $M$  must be equal. So,  $\cos n\theta$  is equal to the  $(1, 1)$  entry of  $M$  and  $\sin n\theta$  is equal to the  $(1, 2)$  entry of  $M$ . Therefore, we have found explicit polynomials in  $\cos \theta$  and  $\sin \theta$  for  $\cos n\theta$  and  $\sin n\theta$ .  $\square$

**Remark** It is also possible to prove this lemma using DeMoivre's theorem, which states that  $\cos n\theta + i \sin n\theta = e^{in\theta} = (e^{i\theta})^n = (\cos \theta + i \sin \theta)^n$ . Expanding this expression gives a polynomial in  $\sin \theta$  and  $\cos \theta$  as well.

We can now prove the main result from this paper:

**Theorem 2.3** Let  $K_t : (-\pi, \pi) \rightarrow \mathbb{R}^3$  be a parameterization of a knot for which each component is a rational function in  $\sin \theta$  and  $\cos \theta$ . Then there exists a function  $K_r : \mathbb{R} \rightarrow \mathbb{R}^3$  with the same image as  $K_t$  for which each component is a rational function.

**Proof** By Lemma 2.2, we know that  $K_t$  can be expressed entirely in terms of  $\sin \theta$  and  $\cos \theta$ . By Lemma 2.1, we can transform each  $\sin \theta$  and  $\cos \theta$  into a rational function. Since the composition of rational functions is also a rational function, we have found that the rational parameterization  $K_r$  has the same image as  $K_t$ . As previously noted, the domain of each function can be extended so that the two curves miss no points.  $\square$

This shows that a rational parameterization can be found for any knot which has a trigonometric parameterization. Trautwein's proof of the existence of trigonometric parameterizations for every knot shows that a rational parameterization exists for any knot.

**Remark** A method for creating a rational parameterization of a given knot "from scratch" (that is, from a given diagram of the knot) is suggested in [4]. It seems likely that a direct proof that a rational parameterization can be found for any knot is possible. This would most likely make use of the Stone-Weierstrauss theorem.

### 3 Torus Knots

Knowing that every knot has a trigonometric parameterization is quite different from actually finding such parameterizations. However, it is relatively easy to find trigonometric constructions of torus knots, and therefore to find rational parameterizations as well.

A *Torus knot* is a knot which can be deformed into another knot which is drawn on the surface of a torus. A torus is diffeomorphic to a circle of circles (that is,  $S^1 \times S^1$ ). These two fundamental circles which form a torus are referred to as the *longitudinal* and *meridional* circles.

This allows us to more precisely define a torus knot:

**Definition** Let  $p, q$  be relatively prime positive integers. A  $(p, q)$  *torus knot* is a knot which is equivalent to another torus knot which performs  $p$  complete rotations about the meridian of the torus, and  $q$  complete rotations about the longitude of the torus.

Expressed another way: a torus knot is defined by its behavior as it twists around the torus. If we were to follow a  $(p, q)$  torus knot as it was drawn, we would make  $q$  complete rotations in the direction of the longitudinal circle, and as we were doing this, we would make  $p$  complete rotations in the direction of the meridional circle.

Some of these knots are very familiar. For example, the  $(3, 2)$  torus knot is actually the trefoil. Note also that any  $(1, n)$  or  $(n, 1)$  torus knot is equivalent to the unknot.

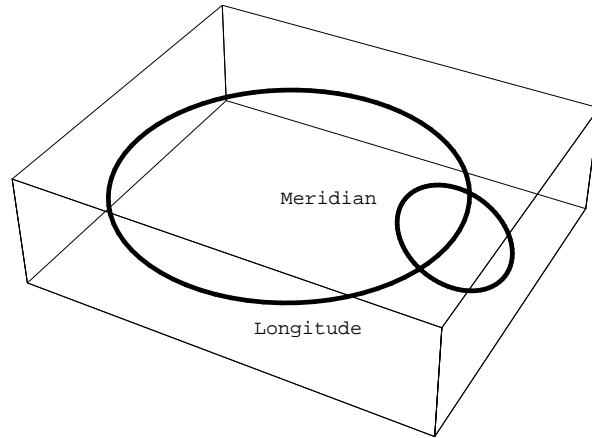


Figure 1: The fundamental circles of the torus.

In order to find knots on the torus, it is important first to know how to construct a torus. We will use this parameterization of a torus, which is constructed entirely from trigonometric functions:

$$\Phi(r, s) = (\cos r(3 + \cos s), \sin r(3 + \cos s), \sin s)$$

where both  $r$  and  $s$  range from 0 to  $2\pi$ .

Here, the parameter  $r$  affects the way that the torus is drawn in the longitudinal direction, and the parameter  $s$  affects the way that the torus is drawn in the meridional direction. The longitudinal circle of this parameterization has radius 3, and the meridional circle has radius 1. These are arbitrary sizes, but they make a well-proportioned torus for our further work. All torus knots in this paper will be equivalent to torus knots on the surface of this standard embedding of a torus in  $\mathbb{R}^3$ .

## 4 Tying the knot

Torus knots are a lovely class of knots, but how can we construct them?

The 2-dimensional torus embedded in 3 dimensions is represented by a function of 2 parameters and 3 components. We want to find a 1-dimensional line on the surface of the torus which wraps around the torus  $p$  times in the meridional direction, and  $q$  times in the longitudinal direction. This means that we need a function of 1 parameter and 3 components. To derive the parametric equations for this line, we will find a way to express both  $r$  and  $s$  in terms of a single new parameter,  $\theta$ .

It will be useful to note that a circle parameterized by:

$$\begin{aligned} x(\theta) &= \cos n\theta \\ y(\theta) &= \sin n\theta \end{aligned}$$

is drawn over itself a total of  $n$  times for  $0 \leq \theta < 2\pi$ .

First, consider the parameter  $r$ , which affects the drawing of the torus in the longitudinal direction. We want our knot to wrap around the torus  $q$  times in this direction. As  $r$  takes values in  $[0, 2\pi)$ , each longitudinal circle of the trigonometric torus is drawn exactly once. If we substitute  $r = q\theta$ , each longitudinal circle will be drawn  $q$  times as  $\theta$  ranges from 0 to  $2\pi$ , which is precisely the way that we want our torus knot to work.

Similarly, we want our knot to wrap around the torus  $p$  times in the meridional direction, which is controlled by  $s$ . We can substitute  $s = p\theta$  to assure that each meridional circle is drawn  $p$  times.

Substituting these changes in to  $\Phi(r, s)$  to get  $\Phi(q\theta, p\theta)$ , we find that the general parameterization of a  $(p, q)$  torus knot in terms of trigonometric functions is:

$$\Phi(q\theta, p\theta) = (\cos q\theta(3 + \cos p\theta), \sin q\theta(3 + \cos p\theta), \sin p\theta)$$

for  $0 \leq \theta < 2\pi$ . It is possible to create any  $(p, q)$  torus knot in this fashion.

Once we have found the trigonometric parameterization of a  $(p, q)$  torus knot in terms of  $\cos n\theta$  and  $\sin n\theta$ , Lemma 2.2 allows us to express the trigonometric functions as polynomials in  $\cos \theta$  and  $\sin \theta$ . Making a change of parameters  $\alpha = \theta - \pi$  to account for our slightly different domains allows us to change each  $\sin \alpha$  and  $\cos \alpha$  into a rational function, making the entire parameterization of the torus knot into a rational parameterization by Theorem 2.3.

This shows that every torus knot can be represented in terms of rational functions.

**Example** We will construct the  $(3, 2)$  torus knot from trigonometric functions, and transform those into rational functions. As previously mentioned, this torus knot is equivalent to the trefoil.

Using the formula derived above, we know that the trigonometric form of the  $(3, 2)$  torus knot is:

$$\Phi(3\theta, 2\theta) = (\cos 2\theta(3 + \cos 3\theta), \sin 2\theta(3 + \cos 3\theta), \sin 3\theta)$$

Using the method described in Lemma 2.2, we can find expressions for  $\sin 3\theta$ ,  $\cos 3\theta$ ,  $\sin 2\theta$ , and  $\cos 2\theta$ :

$$\begin{aligned} \sin 3\theta &= 2 \cos^2 \theta \sin \theta + \sin \theta (\cos^2 \theta - \sin^2 \theta) \\ \cos 3\theta &= -2 \cos \theta \sin^2 \theta + \cos \theta (\cos^2 \theta - \sin^2 \theta) \\ \sin 2\theta &= 2 \cos \theta \sin \theta \\ \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \end{aligned}$$

Substituting these expressions in to  $\Phi(3\theta, 2\theta)$ , we get this very ugly result:

$$\begin{aligned} \Phi(3\theta, 2\theta) &= \\ &((-2 \cos \theta \sin^2 \theta + \cos \theta (\cos^2 \theta - \sin^2 \theta))(3 + \cos^2 \theta - \sin^2 \theta), \end{aligned}$$

$$(2 \cos \theta \sin \theta)(3 + -2 \cos \theta \sin^2 \theta + \cos \theta (\cos^2 \theta - \sin^2 \theta)),$$

$$2 \cos^2 \theta \sin \theta + \sin \theta (\cos^2 \theta - \sin^2 \theta))$$

Finally, we can make rational substitutions  $\sin \theta = \frac{2t}{1+t^2}$  and  $\cos \theta = \frac{1-t^2}{1+t^2}$  as in Lemma 2.1. Simplifying, this gives us a rational parameterization of the (3, 2) torus knot:

$$\Phi(t) = \left( \frac{2(1 - 6t^2 + t^4)(2 - 3t^2 + 12t^4 + t^6)}{(1 + t^2)^5}, -\frac{8t(-1 + t^2)(2 - 3t^2 + 12t^4 + t^6)}{(1 + t^2)^5}, \frac{6t - 20t^3 + 6t^5}{(1 + t^2)^3} \right)$$

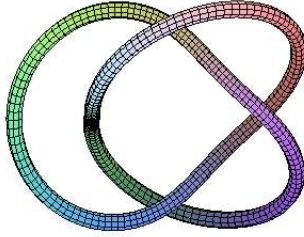


Figure 2: A (3, 2) torus knot is a trefoil.

for  $t$  in  $\mathbb{R}$ . If we add the “point at infinity” to make the domain  $\mathbb{R} \cup \infty$ , we do not miss any points on this knot.

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