ALGORITHM FOR CONSTRUCTING COMPACT RATIONAL KNOTS

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Abstract. This paper defines an algorithm for constructing compact rational knots for a given knot-type. A trefoil following this algorithm is constructed. Conjectures on the degree necessary for this algorithm to work are stated. This research is supported by the NSF, Alan Durfee and Donal O'Shea gave advice and guidance throughout this research project, and David Clark, Virginia Peterson, Craig Phillips, and Alexandra Zusser also participated in this research.

1. Introduction

First, this paper defines an algorithm for constructing compact rational knots for a given knot-type. Then we construct a trefoil following this algorithm.

2. Algorithm for Constructing Compact Rational Knots for a Given Knot-Type

(1) Draw knot in Dowker form, which simply means that all crossings occur on the x-axis, and the crossings are labelled in order from 1 to 2n.
(2) Choose t_k such that $|t_k - t_{k+1}| = a$, $\forall k$, and for some $a \in \mathbb{R}$

$$\begin{cases} \text{if } k \leq n, \text{let } t_k < 0 \\ \text{if } k \geq n+1, \text{let } t_k > k \end{cases}$$

(3) Let $g(t) = \frac{g_1(t)}{g_2(t)}$ and $f(t) = \frac{f_1(t)}{f_2(t)}$, where $g_i(t)$ and $f_i(t)$ are polynomials, and neither $g_2(t)$ nor $f_2(2)$ have real roots.
(4) Choose sufficient degrees for $g(t)$ and $f(t)$, $(\frac{4n}{4n})$ and $(\frac{4n-1}{4n})$, respectively.
(5) Construct $g(t)$ such that $g(t_k) = 0$, $\forall$ k. Depending on the knot-type it may be necessary for $g(t)$ to have roots between a particular t_i and t_{i+1} in order to avoid extra crossings. (Note: This will be obvious once the knot is put into Dowker form).
(6) Let $g_2(t) = \sum_{k=0}^{2n} b_{2k} t^{2k}$, where b_{2k} are chosen such that on the interval $[t_{-n}, t_n]$, $|g(t)| \leq 0.5$, and $\lim_{t\to\infty} g(t) = 1$. This will allow us to construct $f(t)$ such that the trace of $(f(t), g(t))$ will not having extra crossings.
(7) Let $f_1(t) = \sum_{k=0}^{4n-1} c_k t^k$, and let $f_2(t) = \sum_{k=0}^{4n} d_k t^k$.
(8) Let $f(t_i) = f(t_j) = m_{ij}$, where t_i and t_j are values that yield double points, and each m_{ij} is chosen such that the dowker form will be symmetric.
(9) Choose the coefficients of $f(t)$ such that they lie in the solution space of the following set of inequalities:

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$$\begin{cases} f(t_i) = f(t_j) = m_{ij} \\ f_2(t) \neq 0 \ \forall \ t \\ \text{if } g(t) = g(s), t \neq t_i, \text{ and } s \neq t_j, \text{ then } f(t) \neq f(s) \end{cases}$$

(10) Now $(f(t), g(t))$ gives a trace of the desired knot.
(11) Construct $h(t)$ such that $\lim_{t\to\infty} h(t) = 0$. Also, if in $[t_{i-1}, t_i]$ we have only under crossing or only over crossings and in $[t_i, t_{i+1}]$ we have the other form of crossing, then we define

$$h(t) = \frac{\pm \prod_{i=1}^{2n+1}(t - t_i)}{1 + t^{2n+2}}$$

The sign of $h(t)$ is depends on the first crossing from the left [1].

Remark Step 4 of the above algorithm is a conjecture. It has not been shown that degrees $\frac{4n}{4n}$ and $\frac{4n-1}{4n}$ are large enough to parameterize a compact knot without additional crossings. It is known that the degree of the numerator must be at least $2n$ to allow for the crossings to occur on the x-axis. If we allow for zeros between the crossings, we could have at most $n+1$ additional zeros. $\frac{4n}{4n}$ and $\frac{4n-1}{4n}$ are large enough for this, however, there may be additional invariants of the knot that require the degree to be higher. So, it may be necessary to use a degree than $\frac{4n}{4n}$.

3. Algorithm Applied to the Trefoil

Let $t_1 = -3, t_2 = -2, t_1 - 1, t_4 = 1, t_5 = 2$, and $t_6 = 3$.
Let $g(t)$ be defined by:

$$g(t) = \frac{(t+3)(t+1)(t+2)(t-1)(t-2)(t-3)}{100 + t^2 + t^4 + t^6}$$

Let $f(t_1) = -1, f(t_2) = 0, f(t_3) = 1, f(t_4) = -1, f(t_5) = 0$, and $f(t_6) = 1$

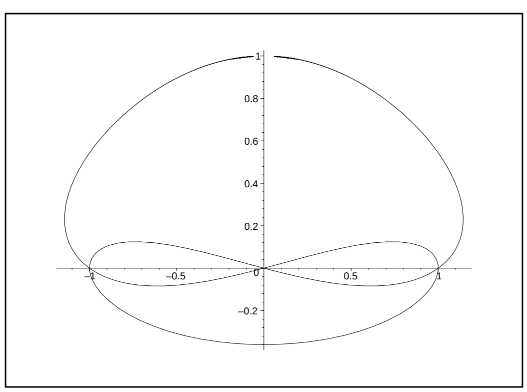

FIGURE 1. Trace of Dowker Trefoil Constructed Using Algorithm

After solving for the coefficients of $f(t)$, we have:

$$f(t) = \frac{(6t^5 - 8t^3 - 64t)}{t^6 + \frac{7}{9}t^4 + \frac{218}{9}t^2 + 40}$$

Then we define $h(t)$ by:

$$h(t) = \frac{-(t+2.5)(t+1.5)t(t-1.5)(t-2.5)(t-3.5)}{1+t^8}$$

Now we have a Dowker Trefoil (See Figure 1).

4. Conjectures

- $\frac{2^n}{2^n}$ is sufficiently large to construct the trace of a knot with n crossings.
- $\frac{4n}{4n}$ and $\frac{4n-1}{4n}$ are sufficiently large to construct the trace of a knot with n crossings.

References

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