ON THE DEGREES OF RATIONAL KNOTS

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Abstract. In this paper, we explore the issue of minimizing the degrees on rational knots. We set a bound on these degrees using Bézout's theorem, define minimal degree sequences for rational functions, and explore the degree sequences of the trefoil and figure-eight knots. This research was conducted at the Mount Holyoke College REU program during the summer of 2002. Special thanks go out to our advisors, Alan Durfee and Donal O'Shea, and the other members of our research group, David Clark, Virginia Peterson and Craig Phillips.

1. Introduction

Obtaining rational parameterizations of knots, whether by converting polynomial [5, 6] or trigonometric [2] parameterizations to rational ones or by constructing rational parameterizations from scratch [4, 6], yields functions with very high degrees on the polynomials in the numerator and denominator. In many cases, these degrees can be reduced to as little as one fourth of their original size. This phenomenon leads to the question,"what are the minimum degrees with which a knot type can be parameterized rationally?" Having the minimum degrees of knot types would also be useful in trying to classify rational knots according to their degrees.

2. Bounds on Rational Knots Based on Degree

Let us define the degree of a rational function by the following:

Definition 2.1. *The Degree of a rational function $f(t) = \frac{p(t)}{q(t)}$, where $p(t)$ and $q(t)$ are relatively prime polynomials of degree p and degree q is defined to be $\frac{p}{q}$*

Let $\alpha(t) = (f(x), g(x), h(x))$ be a rational function and denote by $(\frac{m}{n}, \frac{p}{q}, \frac{r}{s})$ the degrees of $(x(t), y(t), z(t))$. If $\alpha(t)$ parameterizes a knot, then we can find an upper bound on its crossing number by considering the degrees of $x(t)$, $y(t)$ and $z(t)$.

Theorem 2.1. *Let c be the crossing number of a rationally parameterized knot and $(\frac{m}{n}, \frac{p}{q}, \frac{r}{s})$ be the degrees of its components. Then $c \leq \frac{1}{2}(m + n - 1)(p + q - 1)$.*

Proof. Since the crossing number of a knot, c, is the least number of crossings, or self-intersections, in any projection of the knot [1, page 67], we can put an upper bound on c by considering the $x - y$, $x - z$, and $y - z$ projections. Let us first consider the $x - y$ projection. Without loss of generality, we can assume that

$$x(t) = \frac{t^m + ...}{t^n + ...},$$

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$$y(t) = \frac{t^p + \dots}{t^q + \dots},$$

where "..." denotes lower-order terms. A self-intersection occurs when there exist t_1 and t_2 such that $t_1 \neq t_2$ and $x(t_1) = x(t_2)$, $y(t_1) = y(t_2)$. This gives us the equations

$$\frac{t_1^m + \dots}{t_1^n + \dots} = \frac{t_2^m + \dots}{t_2^n + \dots} \text{ and } \frac{t_1^p + \dots}{t_1^q + \dots} = \frac{t_2^p + \dots}{t_2^q + \dots}$$

for x and y, respectively. By cross-multiplying each of the above equations and collecting all terms on the left hand side, we get a polynomial of the form

$$t_1^m t_2^n - t_1^n t_2^m + \dots = 0$$

for x, and a polynomial of the form

$$t_1^p t_2^q - t_1^q t_2^p + \dots = 0$$

for y. These are polynomials in degree $m+n$ and degree $p+q$, respectively. $t_1 = t_2$ is a solution to both of these equations, but an unwanted one. To remove it, we can divide each of them by $t_1 - t_2$ to obtain

$$\frac{t_1^m t_2^n - t_1^n t_2^m + \dots}{t_1 - t_2} = 0$$

$$\frac{t_1^p t_2^q - t_1^q t_2^p + \dots}{t_1 - t_2} = 0,$$

which are polynomial equations with degrees $m+n-1$ and $p+q-1$, respectively.

By Bézout's theorem, the number of common solutions is at most the product of the degrees of these two polynomials, $(m+n-1)(p+q-1)$ [3, page 91]. Since each crossing of the projection in fact gives two solutions, the number of crossings is at most $\frac{1}{2}(m+n-1)(p+q-1)$. To find the lowest number for the bound, we use the projection with the lowest degrees, i.e. if $(m+n) < (p+q) < (r+s)$, then

$$\frac{(m+n-1)(p+q-1)}{2} < \frac{(m+n-1)(r+s-1)}{2} < \frac{(p+q-1)(r+s-1)}{2},$$

which gives $c \leq \frac{1}{2}(m+n-1)(p+q-1)$ as the most accurate bound. □

A bound of this type is useful for choosing a starting point when trying to determine the minimal degree sequences of a knot, as we will do in Sections 3, 4, and 5.

2.1. **Example.** Let us consider an example where $deg(x(t), y(t), z(t)) = (\frac{2}{2}, \frac{2}{2}, \frac{2}{2})$. Here, $c \leq 4.5$, so any knot parameterized by such a function would have at most four crossings. Thus, a rational function with degrees $(\frac{2}{2}, \frac{2}{2}, \frac{2}{2})$ can only parameterize an unknot, a trefoil or a figure-eight knot, since these are the only knots that have crossing number ≤ 4 [7, § 6]. Unfortunately, as will be proven in section ??, it is actually impossible to parameterize a trefoil or a figure-eight knot with degrees as low as $(\frac{2}{2}, \frac{2}{2}, \frac{2}{2})$. This means that the only knot with degree $(\frac{2}{2}, \frac{2}{2}, \frac{2}{2})$ is the unknot, suggesting that this method sets a rather high bound on the crossing number. Making the bound lower and more accurate would be a good goal for future research.

3. The Minimal Degree Sequence of a Compact Rational Parameterization

Definition 3.1. *A triple* $(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}) \in \mathbb{Q}^3$ *is said to be a degree sequence for a knot-type K if*

(1) $\frac{p_i}{q_i} \leq 1 \forall i$
(2) $q_1 \leq q_2 \leq q_3$
(3) *If* $q_i = q_{i+1}$*, then* $p_i \leq p_{i+1}$
(4) *there exist real rational functions* $f(t), g(t)$*, and* $h(t)$ *of degree* $\frac{p_1}{q_1}, \frac{p_2}{q_2}$*, and* $\frac{p_3}{q_3}$ *respectively such that the embedding* $t \to (f(t), g(t), h(t))$ *represents K.*

Note: $q_i \in 2\mathbb{N}$.

Definition 3.2. *A degree sequence* $(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}) \in \mathbb{Q}^3$ *is said to be minimal for a knot-type K if for any other degree sequence,* $(\frac{m_1}{n_1}, \frac{m_2}{n_2}, \frac{m_3}{n_3})$*, for K, then* $(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}) \leq (\frac{m_1}{n_1}, \frac{m_2}{n_2}, \frac{m_3}{n_3})$*. Here '*$\leq$*' is the lexicographic ordering in* $\mathbb{N}^6$*, on the sextuplets* $(q_1, q_2, q_3, p_1, p_2, p_3)$ *and* $(n_1, n_2, n_3, m_1, m_2, m_3)$*.*

Example 3.1. $(\frac{1}{4}, \frac{2}{4}, \frac{3}{6}) < (\frac{1}{4}, \frac{3}{4}, \frac{2}{6})$

Example 3.2. $(\frac{1}{4}, \frac{3}{4}, \frac{3}{4}) < (\frac{1}{4}, \frac{2}{4}, \frac{2}{6})$

4. The Minimal Degree Sequence of the Compact Rational Trefoil

According to Bézout's theorem, the first degree sequence that can produce three crossings is $(\frac{1}{2}, \frac{2}{2})$. The $x-y$ projection of the trefoil, $(f(t), g(t))$, has three double points. That is, there must be $t_1 < t_2 < t_3 < t_4 < t_5 < t_6)$ such that:

$$f(t_i) = f(t_{i+3})$$

$$g(t_i) = g(t_{i+3})$$

$$i = 1, 2, 3$$

In order that $(f(t), g(t), h(t))$ trace out a trefoil, we need

$$h(t_1) < h(t_4)$$

$$(\spadesuit) h(t_2) > h(t_5)$$

$$h(t_3) < h(t_6)$$

So, if the degree sequence $(\frac{1}{2}, \frac{2}{2})$ can produce the $x-y$ projection of the trefoil, then it is possible to arrange the 3 double points on the graphs of f and g. We ask whether we can choose $t_1 < t_2 < t_3 < t_4 < t_5 < t_6)$ on the t-axis so that h satisfies $(\spadesuit)$ if h has the following graphs.

The first three figures and their reflections show the only possible forms of a degree $\frac{2}{2}$ rational function. It is clear that this requires four or more monotone regions. Therefore, the trefoil cannot be constructed using a rational function of degree $\frac{2}{2}$ or lower. The fourth figure is the simplest graph on which the three double points of the trefoil can be arranged. It is easy to show that it cannot be formed by a rational function of degree $\frac{2}{2}$ or lower. In fact, with a little work it can be shown that $\frac{2}{4}$ is the lowest degree capable of producing such a graph. Therefore, $(\frac{2}{4}, \frac{2}{4})$ is the first possible minimal degree sequence for the $x-y$ projection of the trefoil.

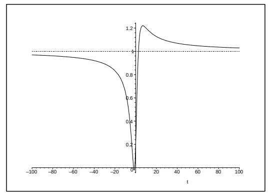

FIGURE 1. A Degree $\frac{2}{2}$ rational function

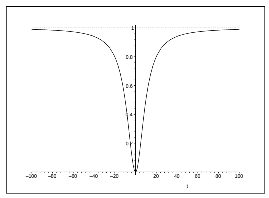

FIGURE 2. A Degree $\frac{2}{2}$ rational function

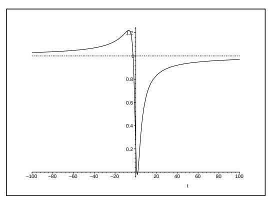

FIGURE 3. A Degree $\frac{2}{2}$ rational function

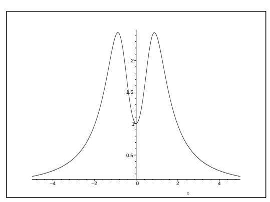

FIGURE 4. Simplest Graph for the Trefoil's Double Points

We define $f_1(t)$ and $g_1(t)$ as follows:

$$f_1(t) = \frac{5t^2 + 2t + 1}{1 + t^2 + t^3 + t^4}$$

$$g_1(t) = \frac{4t^2 + 1}{1 + .1t^2 + t^4}$$

Figure 4 shows the trace of $(f_1(t), g_1(t))$, which is clearly a trefoil projection.

It seems that $(\frac{2}{4}, \frac{2}{4}, \frac{2}{4})$ is the minimal degree sequence for the trefoil. However, an $h(t)$ of degree $\frac{2}{4}$ has not been found that will satisfy ($\spadesuit$) for the given $x - y$ projection. So, we can only conclude that the minimal degree sequence for the compact rational trefoil is greater than $(\frac{2}{4}, \frac{2}{4}, \frac{2}{4})$.

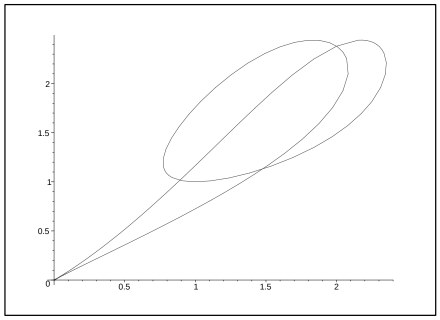

FIGURE 5. Trace of $(f_1(t), g_1(t))$

On the other hand, let us show that it is possible to parameterize a trefoil by a $(\frac{3}{4}, \frac{3}{4}, \frac{3}{4})$ degree sequence. This will give us an upper bound on the minimal degree sequence for a trefoil. Let $\psi_2 : \mathbf{C} \to \mathbf{C}^3$ be defined by:

$$f_2(t) = \frac{t^3 - 3t}{1 + t^4}$$

$$g_2(t) = \frac{(t + 1.523)(t + .04)(t - 2)}{100 + t^4}$$

$$h_2(t) = \frac{(t + 1.8)(t - .1)(t - 1.8)}{10 + t^4}$$

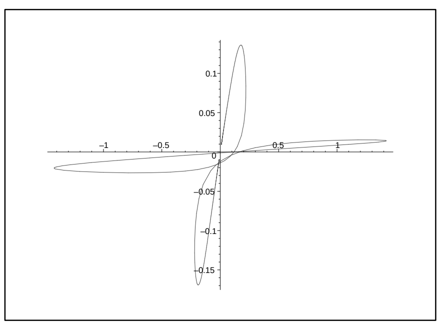

FIGURE 6. $x - y$ projection of $(f_2(t), g_2(t), h_2(t))$

A series of Reidemeister moves can transform this trefoil into one that resembles the standard trefoil. We conclude that the minimal degree sequence of the compact rational trefoil is greater than or equal to $(\frac{2}{4}, \frac{2}{4}, \frac{2}{4})$ and less than or equal to $(\frac{3}{4}, \frac{3}{4}, \frac{3}{4})$.

5. The Minimal Degree Sequence of the Compact Rational figure-eight Knot

According to Bézout's theorem, the first degree sequence that can produce four crossings is $(\frac{2}{2}\ ,\frac{2}{2})$. However, it seems intuitive that the minimal degree sequence for the figure-eight knot would be greater than or equal to that of the trefoil. In order that $(f(t), g(t), h(t))$ trace out a figure-eight knot, we need

$$h(t_1) < h(t_6)$$

$$(\clubsuit) h(t_2) > h(t_5)$$

$$h(t_3) < h(t_8)$$

$$h(t_4) < h(t_7)$$

.

Therefore, the first possible degree sequence for the figure-eight is the same as the trefoil, $(\frac{2}{4}, \frac{2}{4}, \frac{2}{4})$.

Unfortunately, we cannot determine whether one can parameterize the figure-eight knot with a $(\frac{2}{4}, \frac{2}{4}, \frac{2}{4})$ degree sequence. However, an upper bound for the degree sequence has been determined. It is possible to parameterize the figure-eight knot by a $(\frac{3}{4}, \frac{3}{4}, \frac{3}{4})$ degree sequence. Let $\psi_3 : \mathbf{C} \to \mathbf{C}^3$ be defined by:

$$f_3(t) = \frac{t^3 - 3t + 1}{50 + t^4}$$

$$g_3(t) = \frac{(t + 1.83)(t - .14)(t - .65)}{.1 + t^4}$$

$$h_3(t) = \frac{(t + 2)(t - .23)(t - .65)}{1 + t^4}$$

Once again, a series of Reidemeister moves can transform this figure-eight knot into one that resembles the standard one.

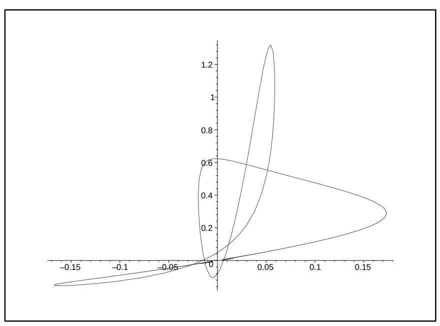

FIGURE 7. XZ projection of $(f_3(t), g_3(t), h_3(t))$

We conclude that the minimal degree sequence of the compact rational figure-eight is greater than or equal to $(\frac{2}{4}, \frac{2}{4}, \frac{2}{4})$ and less than or equal to $(\frac{3}{4}, \frac{3}{4}, \frac{3}{4})$.

6. Conjectures

- The minimal degree sequence of the compact rational trefoil is $(\frac{2}{4}, \frac{2}{4}, \frac{2}{4})$,
- The minimal degree sequence of the compact rational figure-eight is strictly less than $(\frac{3}{4}, \frac{3}{4}, \frac{3}{4})$,
- There exists a condition stronger than Bézout's theorem for reducing degree sequences for a knot with n crossings.

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