THE MINIMAL DEGREE SEQUENCE OF THE POLYNOMIAL FIGURE EIGHT KNOT

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Abstract. This paper defines the minimal degree sequence of a polynomial knot, and determines it for the figure eight knot. The difference between using lexicographic and reverse lexicographic ordering is looked at in the specific case of the figure eight knot. It is determined that these two orderings produce distinct minimal degree sequences for the figure eight know.

1. Introduction

The study of polynomial parameterizations of knots is motivated by a desire to classify knots. Polynomial knots can easily be classified by the degree of their parameterization. However, for a given knot type there exist an infinite number of parameterizations. Therefore, it is useful to classify a knot by its minimal degree parameterization. This paper together with Shastri's paper determines the minimal degree sequence in which the figure eight knot can be parameterized [2].

2. The Minimal Degree of a Polynomial Parameterization

One definition of minimal degree that we will use comes from Mishra's paper[1].

Definition 2.1. *A triple* $(d_1, d_2, d_3) \in \mathbb{N}^3$ *is said to be a degree sequence for a knot-type K if*

1. $d_1 < d_2 < d_3$
2. d_1, d_2, d_3 *are such that none of them lie in the semi-group generated by the other two, and*
3. *there exist real polynomials* $f(t), g(t)$, *and* $h(t)$ *of degree* d_1, d_2, *and* d_3 *respectively such that the embedding* $t \to (f(t), g(t), h(t))$ *represents K.*

Definition 2.2. *A degree sequence* $(d_1, d_2, d_3) \in \mathbb{N}^3$ *is said to be minimal for a knot-type K if for any other degree sequence,* (e_1, e_2, e_3), *for K ,then* $(d_1, d_2, d_3) \leq (e_1, e_2, e_3)$. *Here '*$\leq$*' is the lexicographic ordering in* $\mathbb{N}^3$.

Example 2.1. $(3, 5, 7) < (4, 5, 6)$

We will also use an altered version of Mishra's definition.

Definition 2.3. *A degree sequence* $(d_1, d_2, d_3) \in \mathbb{N}^3$ *is said to be minimal for a knot-type K if for any other degree sequence,* (e_1, e_2, e_3), *for K ,then* $(d_1, d_2, d_3) \leq (e_1, e_2, e_3)$. *Here '*$\leq$*' is the* ***reverse*** *lexicographic ordering in* $\mathbb{N}^3$.

Example 2.2. $(4, 5, 6) < (3, 5, 7)$

1991 *Mathematics Subject Classification.* Primary 11S40, 11G07.
Key words and phrases. Polynomial Knots, Minimal Degree, Figure Eight Knots.
This research is supported by the NSF, grant DMS-9732228.

It is useful to know the minimal degree sequence of a knot using both definitions.

3. The Polynomial Figure Eight Knot

3.1. The Minimal Lexicographic Degree Sequence. From Bézout's theorem we know that the lexicographic minimal degree sequence of any curve with four crossings is of the form $(3, 5, d_3)$. Furthermore, we know that $d_3 \neq 5$ and $d_3 \neq 6$, since 5 and 6 lie in the semi-group generated by 3 and 5. Therefore, $(3, 5, 7)$ is the first degree sequence we need to consider.

In his paper, Shastri constructs an example of a $(3, 5, 7)$ figure eight knot(see Figure 3.1). First he defines $\psi : \mathbf{C} \to \mathbf{C}^3$ by:

$$f(t) = t^3 - 3t$$

$$g(t) = t^5 - 5t^3 + 4t$$

$$h(t) = t^7 - 42t$$

Shastri proves that this mapping defines an imbedding of $\mathbf{C}$ in $\mathbf{C}^3$, by showing that the ring homomorphism $\varphi : k[X, Y, Z] \to k[t]$ defined by:

$$X \to f(t), Y \to g(t), Z \to h(t)$$

is surjective. Also, the XY-projection of φ is an immersion since the derivatives of f and g have no common zeros.

It is a simple calculation to find the t-values that give the four double points $t_1 \neq t_2$ in $\mathbb{R}$ such that $f(t_1) = f(t_2)$ and $g(t_1) = g(t_2)$. The t-values that yield double points are:

$$(t_1, t_6) = (-2, 1)$$

$$(t_2, t_5) = (-(\sqrt{6} + \sqrt{2})/2, (\sqrt{6} - \sqrt{2})/2)$$

$$(t_3, t_8) = (-1, 2)$$

$$(t_4, t_7) = ((\sqrt{2} - \sqrt{6})/2, (\sqrt{2} + \sqrt{6})/2)$$

We note that

$$t_i < t_{i+1}.$$

All that remains is to show that $h(t)$ makes crossings over and under as required. It is easy to check that $h(t)$ satisfies the following inequalities:

$$h(t_1) < h(t_6)$$

$$h(t_2) > h(t_5)$$

$$h(t_3) < h(t_8)$$

$$h(t_4) > h(t_7)$$

Therefore, φ represents the figure eight knot as claimed, and $(3, 5, 7)$ is the minimal lexicographic degree sequence.

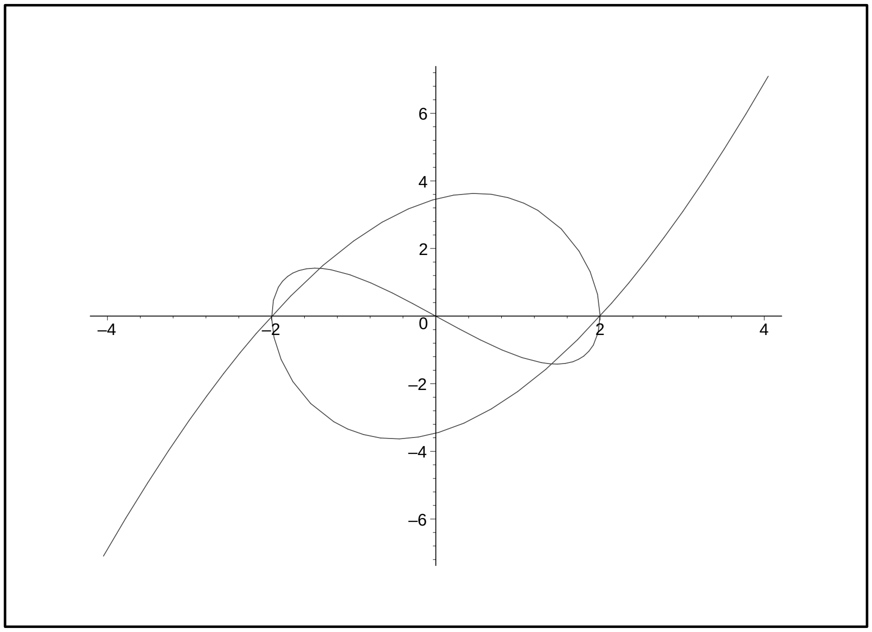

FIGURE 1. Shastri's $(3, 5, 7)$ Figure Eight XY Projection

3.2. **The Minimal Reverse Lexicographic Degree Sequence.** If the minimal reverse lexicographic degree sequence is distinct from $(3, 5, 7)$, then $d_3 < 7$. However, in order to have four crossings we must have $d_2 \geq 5$. Since $d_3 \neq d_2$, we have $d_3 = 6$ and $d_2 = 5$. Bézout's theorem dictates that $d_1 \geq 3$; however, if $d_1 = 3$ then d_3 is in the semigroup generated by d_1. Therefore, $d_1 = 4$ is our only option. Then the only possible minimal reverse lexicographic degree sequence distinct from $(3, 5, 7)$ is $(4, 5, 6)$.

Note: The following construction is intended to show that a $(4, 5, 6)$ degree sequence figure eight knot exists. It is not intended to provide a "nice" mapping.

First, a degree four polynomial that is "similar" to Shastri's degree three polynomial in the interval that the double points occurred is constructed(see Figure 3.2).

$$f_2(t) = -t^4 + 2.279283653t^3 + 5t^2 - 8.63068748t + .35140383$$

Then using the same $g(t)$ as Shastri, we have an XY projection of the figure eight knot. With the aid of a computer it is easy to check that the derivatives of $f_2(t)$ and $g(t)$ have no common zeros, so the XY projection is an immersion.

Then, using a computer, we calculate the t-values that yield double points which are:

$$(t_1, t_6) = (-2.090568217, 1.379221313)$$

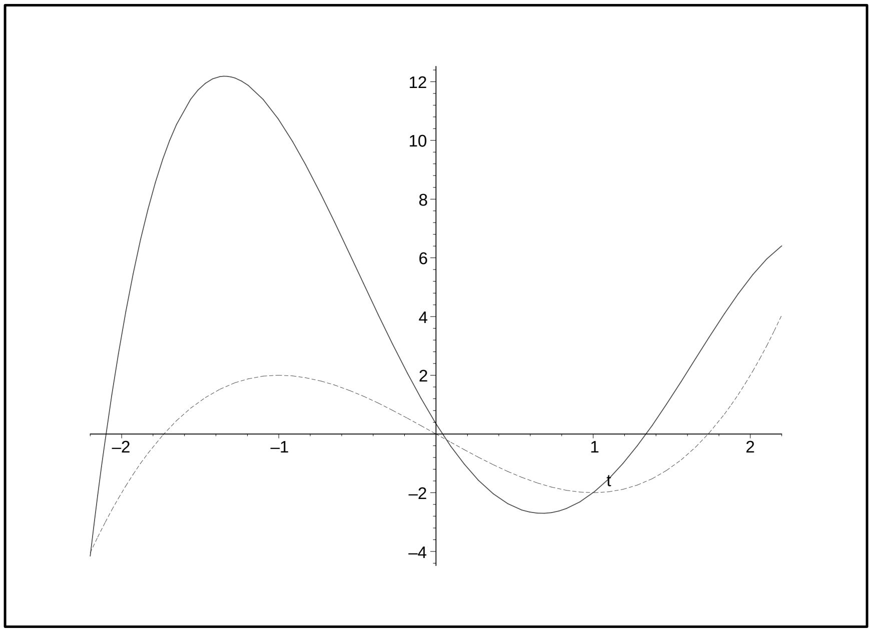

Figure 2. Shastri's Degree 3 and New Degree 4

$$(t_2, t_5) = (-2.031308777, -.2122155248)$$

$$(t_3, t_8) = (-1.916737670, 2.061434333)$$

$$(t_4, t_7) = (-.4386844491, 1.935518580)$$

We note again that

$$t_i < t_{i+1}.$$

Finally, we construct $h_2(t)$ such that it is a degree six polynomial that satisfies the following equations:

$$h(t_1) < h(t_6)$$

$$h(t_2) > h(t_5)$$

$$h(t_3) < h(t_8)$$

$$h(t_4) > h(t_7)$$

The end result is a $(4, 5, 6)$ mapping, $\psi : \mathbf{C} \to \mathbf{C}^3$ defined by:

$$f_2(t) = -t^4 + 2.279283653t^3 + 5t^2 - 8.63068748t + .35140383$$

$$g(t) = t^5 - 5t^3 + 4t$$

$$h_2(t) = (t + 2.06)(t + 1.916737670)(t + .2122155248)(t - 1.379221313)(t - 2.05)(t + 10),$$

that represents the figure eight knot(see Figure 3.2). Therefore, $(4, 5, 6)$ is the minimal reverse lexicographic degree sequence for the figure eight knot.

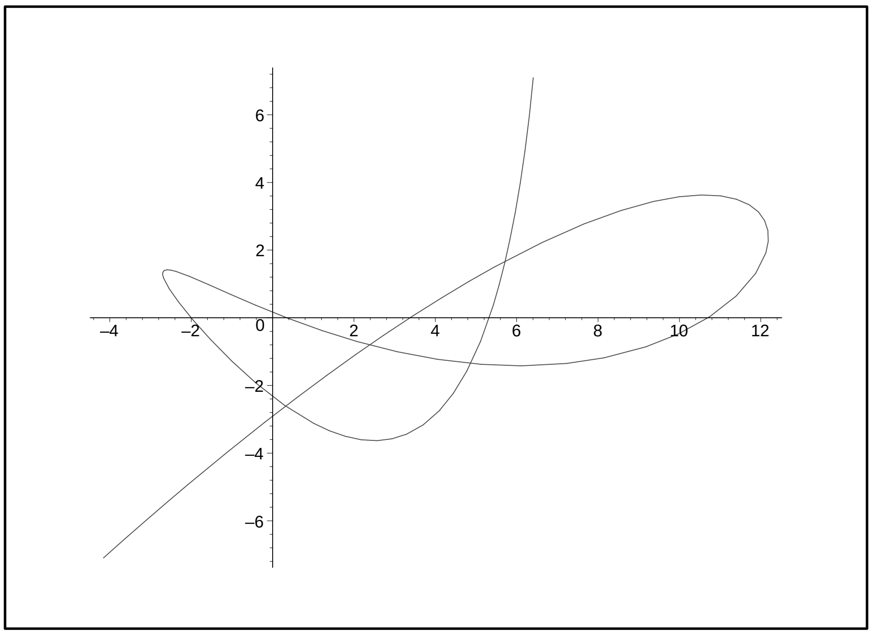

FIGURE 3. (4,5,6) Figure Eight XY Projection

4. CONJECTURES

- If a knot has a minimal lexicographic degree sequence of the form
$$(n, n + 2, n + 4),$$
then its minimal reverse lexicographic degree sequence is
$$(n + 1, n + 2, n + 3).$$
- If a knot has a minimal lexicographic degree sequence of the form
$$(n, n + 2k, n + 4k),$$
where $k \in \mathbb{N}$, then its minimal reverse lexicographic degree sequence is
$$(n + k, n + 2k, n + 3k).$$

- If (d_1, d_2, d_3) is the minimal lexicographic degree sequence of a knot and (e_1, e_2, e_3) is the minimal reverse lexicographic degree sequence of the same knot, then

$$d_1 + d_2 + d_3 = e_1 + e_2 + e_3$$

References

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[2] Anat R. Shastri, *Polynomial Representations of Knots*,*Tôhoku* Math. J. 44(1992), 11-17.

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