COMPLETING A POLYNOMIAL KNOT FROM A PROJECTION
DRAFT 3

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Abstract. Given a knot-type and a projection of it parameterized by two polynomials we can find a third polynomial that will form a representation of the knot in $\mathbb{R}^3$ along with the two given polynomials. This paper discusses methods with which to find this third polynomial and some related results.

1. Preamble

This draft is a work-in-progress. Most of this work was done during the summer of 2004 at the Mt. Holyoke College REU program.

2. Introduction

In his 1991 paper, A. R. Shastri showed that every knot-type can be represented as an embedding of $\mathbb{C}$ in $\mathbb{C}^3$ defined by $t \mapsto (f(t), g(t), h(t))$, where $f(t)$, $g(t)$, and $h(t)$ are polynomials [4]. Several questions naturally follow this discovery. The most immediate of these questions being how one would go about finding these representations, as although Shastri proved his result constructively his general method of construction was not very practical. This paper will deal with methods and some theoretical results relating to the problem of finding a third polynomial to "complete" the knot given two polynomials in a polynomial representation of a knot-type. For discussion on some methods for finding the first two polynomials, see [1].

3. Knot-Types and Polynomial Knots

In this paper we will use Shastri's definitions in [4]. Also recall that a knot type is determined by a knot projection (an embedding of $\mathbb{R}$ in

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$\mathbb{R}^2$ with finitely many crossings, each crossing being an ordinary double point) and a set of under(or over)-crossing data for each crossing to distinguish between knot-types. (See [3] for details.)

We now define polynomial knots.

Definition 3.1. A polynomial knot $\phi(t)$ is the image of a differentiable injection $\phi : \mathbb{R} \to \mathbb{R}^3$ defined by $t \mapsto (f(t), g(t), h(t))$, where $f(t)$, $g(t)$, and $h(t)$ are polynomials with real coefficients.

It is sometimes convenient to denote a polynomial knot $\phi(t) = (f(t), g(t), h(t))$ as the column vector $\phi(\mathbf{t}) = (f(t), g(t), h(t))^T$ or simply as the ordered triple $(f(t), g(t), h(t))$. We will be using these three notations interchangeably.

Let k be a knot-type. As part of his proof that every knot-type has a polynomial representation in $\mathbb{C}^3$, he showed that every knot-type can be represented by an embedding of $\mathbb{R}^1$ in $\mathbb{R}^3$. We will now formally define an association between knot-types and polynomial knots.

Definition 3.2. Let k be a knot-type (a differentiable embedding $k : \mathbb{S}^1 \to \mathbb{S}^3$ such that k preserves the base point, or $k(0,1) = (0,0,0,1)$) and $(f(t), g(t), h(t))$ be a polynomial knot. If $k(t,0) = (f(t), g(t), h(t), 0)$, then we say that $(f(t), g(t), h(t))$ is of knot-type k.

Definition 3.3. The degree of a polynomial knot $\phi(t) = (f(t), g(t), h(t))$ is the maximum of $deg(f(t)$, $deg(g(t))$, and $deg(h(t))$.

4. The Polynomial Knot Completion Problem

In this section, much of the work is done from the point of view of a knot projection defined by setting the last coordinate equal to 0. However, by symmetry, all of the arguments as just as valid for the projections obtained by setting the first or second coordinate to 0.

4.1. The Problem and The Existence of a Solution.

Let k be a knot-type, $f(t)$ and $g(t)$ be known polynomials, and assume that $t \mapsto (f(t), g(t), 0)$ is a knot projection of k. We'll call such a knot projection a polynomial projection. Our goal is to find all polynomials $h(t)$ such that $(f(t), g(t), h(t))$ is of type k. We say that such a $h(t)$ completes a knot of type k under projection $t \mapsto (f(t), g(t), 0)$.

Definition 4.1. Let k be a knot-type and $t \mapsto (f(t), g(t), 0)$ be a polynomial projection of k. We call the ordered pair (u, v) a crossing pair of the projection if $u \neq v$, $f(u) = f(v)$, and $g(u) = g(v)$ and if a polynomial knot $(f(t), g(t), h(t))$ is of knot-type k, $h(u) > h(v)$.

In other words, each pair (u, v) corresponds to a crossing in the knot projection, with u being the over-crossing. Since the knot-type of a knot is determined by a knot projection and its crossing data, we have the following lemma.

Lemma 4.2. *A polynomial $h(t)$ completes a knot of type k under projection $t \mapsto (f(t), g(t), 0)$ if and only if $h(u_i) > h(v_i)$ for each crossing pair (u_i, v_i) of the projection.*

We now use this lemma to give a constructive proof that such an $h(t)$ of a specific degree must exist.

Theorem 4.3. *Let k be a knot-type. Let $t \mapsto (f(t), g(t), 0)$ be a polynomial projection of k and $\{(u_i, v_i)|i = 1, \ldots, c\}$ be the set of all its crossing pairs. There exists a polynomial $h(t)$ such that $(f(t), g(t), h(t))$ is a polynomial knot of type k. Furthermore, $deg(h(t)) \leq 2c - 1$.*

Proof. Since c is finite, we can arrange all the u_i's and v_i's into a finite sequence w_k, in increasing order. Define a new sequence of real polynomials $x_k(t)$ in t as follows: if $w_k = u_i$ for some i and $w_{k+1} = v_j$ for some j, or if $w_k = v_i$ for some i and $w_{k+1} = u_j$ for some j, $x_k(t) = \frac{w_k + w_{k+1}}{2}$; otherwise $x_k(t) = t - 1$. Let $p(t) = \sum(t - x_k)$. It is easy to verify via Lemma 3.2 that the desired $h(t)$ is either $p(t)$ or $-p(t)$ and that $deg(h(t)) \leq 2c - 1$.

An alternate proof can be obtained by constructing $h(t)$ via fitting a polynomial curve through the points $(u_i, 1)$ and $(v_i, 1)$ for $i = 1, \ldots, c$ in $\mathbb{R}^2$. Since there are $2c$ points with distinct x coordinates, the desired polynomial exists and is of degree $2c - 1$ or less. □

4.2. **Generating Other Solutions from a Specific Solution.** The previous theorem showed that it is always possible to construct a polynomial $h(t)$ to complete a knot given a polynomial projection, we now show that from one solution we can generate a class of infinitely many solutions. We will then use this result to find a $h(t)$ with a the minimal number of terms.

Theorem 4.4. *Let $(f(t), g(t), h(t))$ be a polynomial knot of type k. Assume that $t \mapsto (f(t), g(t), 0)$ be a polynomial projection of k and let $C = \{(u_i, v_i)|i = 1, \ldots, c\}$ be the set of all its crossing pairs. If $p(t)$ is a polynomial and $p(u_i) = p(v_i)$ for all $(u_i, v_i) \in C$, then $(f(t), g(t), h(t) + p(t))$ is a polynomial knot of type k.*

Proof. Let $t, s \in \mathbb{R}$ and assume that $t \neq s$. $(f(t), g(t), h(t) + p(t)) = (f(s), g(s), h(s) + p(s))$ implies $f(t) = f(s)$ and $g(t) = g(s)$. However, since $(f(t), g(t), h(t))$ is a polynomial knot $h(t) \neq h(s)$, and hence either (s, t) or (t, s) is a crossing pair of the projection $t \mapsto (f(t), g(t), 0)$.

If (s, t) is a crossing pair. Hence $h(t) < h(s)$ By hypothesis we know that $p(t) = p(s)$, so $h(t)+p(t) < h(s)+p(s)$ and $(f(t), g(t), h(t)+p(t)) < (f(s), g(s), h(s) + p(s))$. If (t, s) is a crossing pair we get the symmetric result $(f(t), g(t), h(t) + p(t)) > (f(s), g(s), h(s) + p(s))$. Therefore $(f(t), g(t), h(t) + p(t))$ is a polynomial knot. Since it has the same crossing data as $(f(t), g(t), h(t))$ under projection $t \mapsto (f(t), g(t), 0)$, it is a polynomial knot of type k. □

Corollary 4.5. *If $(f(t), g(t), h(t))$ is a polynomial knot of type k, then so are $(f(t), g(t), h(t) + a \cdot f(t) + b \cdot g(t)) \forall a, b \in \mathbb{R}$.*

Note that by symmetry, Theorem 3.4 (and as its immediate consequence, Corollary 3.5) can be applied to the other two components as well. Applying Corollary 3.5 to the three components in sequence we obtain the following generalization of Corollary 3.5.

Corollary 4.6. *If $(f(t), g(t), h(t))$ is a polynomial knot of type k, then so are $(f(t) + x_1 \cdot g(t) + x_2 \cdot h(t), g(t) + x_3 \cdot f(t) + x_4 \cdot h(t), h(t) + x_5 \cdot f(t) + x_6 \cdot g(t)) \forall x_1, \ldots, x_6 \in \mathbb{R}$.*

This shows that from any polynomial knot we can generate infinitely many other polynomial knots of the same type. However, it is easy to see that we cannot generate all polynomial knots of the same type using this method, as the degree of the polynomial knot cannot be increased through this method. We present here a degree 7 trefoil, and will present a degree 5 trefoil later in the paper:

$$x(t) = t^3 - 3t$$

$$y(t) = t^2(t^2 - 1)(t^2 - 4)$$

$$z(t) = t^7 - 42t.$$

5. Matrix Representation of the Knot Completion Problem

We have shown that given a knot-type and a polynomial projection with c crossings we can find some polynomial $h(t)$ with degree $2c-1$ or less (similarly for the number of terms) that completes the knot. We will now attempt to see if a simpler solution (one with a lower degree or fewer terms) can be found.

5.1. **Translating the Problem.**

Let us rephrase our problem in terms of matrix inequalities. We begin by defining a matrix of constrains on the coefficients of a solution $h(t)$.

Definition 5.1. Let k be a knot-type. Let $p(t) = (f(t), g(t), 0)$ be a polynomial projection of k and let $C = \{(u_i, v_i) | i = 1, \ldots, c\}$. We call the matrix

$$\begin{pmatrix} u_1^1 - v_1^1 & \cdots & u_1^d - v_1^d \\ \vdots & \ddots & \vdots \\ u_c^1 - v_c^1 & \cdots & u_c^d - v_c^d \end{pmatrix}$$

the degree d crossing pair matrix of the projection $p(t)$.

We can now translate the knot completion problem into a matrix inequality by first translating it into a set of linear inequalities with the coefficients of the solution $h(t)$ as variables.

Lemma 5.2. *Let k be a knot-type. Let $p(t) = (f(t), g(t), 0)$ be a polynomial projection of k. Let $h(t) = x_d t^d + x_{d-1} t^{d-1} + \ldots + x_1 t + x_0$, where $x_d > 0$. Then $h(t)$ completes k under $p(t)$ if and only if $x_d \mathbf{A}\mathbf{x} > 0$, where $\mathbf{A}$ is the degree d crossing pair matrix of $p(t)$ and $\mathbf{x} = (x_1 x_2 \ldots x_{d-1} 1)^T$.*

Proof. As before, By Lemma 3.2 a polynomial $h(t) = x_d t^d + x_{d-1} t^{d-1} + \ldots + x_1 t + x_0$ completes k under projection $p(t)$ if and only if $h(u_i) > h(v_i)$ for $i = 1, \ldots, c$. In other words,

$$x_d \begin{pmatrix} u_1^d - v_1^d & \cdots & u_1^1 - v_1^1 \\ \vdots & \ddots & \vdots \\ u_c^d - v_c^d & \cdots & u_c^1 - v_c^1 \end{pmatrix} \begin{pmatrix} 1 \\ x_{d-1} \\ \vdots \\ x_1 \end{pmatrix} > 0. \tag{5.1}$$

∎

5.2. **Solution With Small Number of Terms.** We now use an elementary result from linear algebra to examine crossing pair matrices to obtain the following theorem.

Theorem 5.3. *Let k be a knot-type and $p(t) = (f(t), g(t), 0)$ be a polynomial projection of k with c crossings. There is a polynomial $h(t)$ with less than c terms that completes k under $p(t)$.*

Proof. By Theorem 3.3, there exists a polynomial $\overline{h}(t)$ of some degree d that completes k under $p(t)$. Let $\mathbf{A} \in \mathbb{R}_{c \times d}$ be the degree d crossing pair matrix for $p(t)$. If $d \leq c$, then $\overline{h}(t)$ has at most d non-constant terms. Let $h(t) = \overline{h}(t) - c(t)$, where $c(t)$ is the constant term of $\overline{h}(t)$. Then $h(t)$ has at most c terms and by Theorem 3.4, $(f(t), g(t), h(t))$ is of the same knot type as $(f(t), g(t), \overline{h}(t)$, so $h(t)$ completes the knot.

Assume $d > c$. Then some $n \leq c$ columns of $\mathbf{A}$ form an ordered basis β for the columns. Now we define $d - n$ new polynomials $p_k(t)$ by

$$p_k(t) = t^k - \sum_{\mathbf{A}_{*,j} \in \beta} a_j t^j$$

where a_j satisfy

$$\mathbf{A}_{*,k} = \sum_{\mathbf{A}_{*,j} \in \beta} a_j \mathbf{A}_{*,j}$$

and $\mathbf{A}_{*,k}$ is not in β. Notice that by our construction, $p_k(s) - p_k(t) = 0$ whenever (s, t) is a crossing pair of projection $p(t)$. We now construct $h(t)$ by adding multiples of $p_k(t)$ to $h(t)$ to eliminate the t^k terms and then subtracting the constant term of $\overline{h}(t)$ from the result. By construction $h(t)$ has at most c terms. By Theorem 3.4, $(f(t), g(t), h(t))$ is of the same knot type as $(f(t), g(t), h(t)$, so $h(t)$ completes the knot. □

5.3. **Existence of Low-Degree Solution.** Let $\epsilon > 0$ and let ϵ be the $1 \times c$ column vector $(\epsilon \ldots \epsilon)^T$. We can rephrase inequality (4.1) into a non-strict inequality as

$$\begin{pmatrix} u_1^d - v_1^d & \cdots & u_1^1 - v_1^1 \\ \vdots & \ddots & \vdots \\ u_c^d - v_c^d & \cdots & u_c^1 - v_c^1 \end{pmatrix} \begin{pmatrix} x_d \\ \vdots \\ x_1 \end{pmatrix} \geq \begin{pmatrix} \epsilon \\ \vdots \\ \epsilon \end{pmatrix} \tag{5.2}$$

or as

$$\begin{pmatrix} u_1^{d-1} - v_1^{d-1} & \cdots & u_1^1 - v_1^1 \\ \vdots & \ddots & \vdots \\ u_c^{d-1} - v_c^{d-1} & \cdots & u_c^1 - v_c^1 \end{pmatrix} \begin{pmatrix} x_{d-1} \\ \vdots \\ x_1 \end{pmatrix} \geq x_d \begin{pmatrix} v_1^d - u_1^d \\ \vdots \\ v_c^d - u_c^d \end{pmatrix} + \begin{pmatrix} \epsilon \\ \vdots \\ \epsilon \end{pmatrix}. \tag{5.3}$$

Now that we are no longer dealing with strict linear inequalities, we can apply a useful theorem from linear programming. [2]

Theorem 5.4. *(Gale's Theorem of Alternatives for Inequalities) Let* $\mathbf{A}$ *be a* $p \times n$ *matrix and* $\mathbf{c} \in \mathbb{R}^p$*, then either* $\mathbf{Ax} \geq \mathbf{c}$ *has a solution, or* $\mathbf{A}^T\mathbf{y} = 0$, $\mathbf{cy} \geq 1$, $\mathbf{y} \geq 0$ *has a solution, but never both.*

Proof. See [2]. □

Using Gale's Theorem and inequalities (4.2) and (4.3), we can obtain results that provide sufficiency conditions of when a solution of degree d to the knot completion problem exists.

Theorem 5.5. *Let k be a knot-type and let $p(t) = (f(t), g(t), 0)$ be a c- crossing polynomial projection of k. Let $\mathbf{A}$ be the degree d crossing pair matrix of $p(t)$. If $\mathbf{A}^T\mathbf{y} = 0$ has no non-zero solution $\mathbf{y}$ then there is a polynomial $h(t)$ of degree $r \leq d$ that completes the knot.*

Proof. Observe that if a solution $\mathbf{x}$ exists for inequality (4.2), then a polynomial $h(t) = x_d t^d + \ldots + x_1 t$ completes the knot by Lemma 3.2. Note that $h(t)$ has degree $r \leq d$ (where r is the largest index of the non-zero elements of $\mathbf{x}$–note that r must be positive as since $\epsilon \neq 0$, $\mathbf{x} \neq 0$).

Applying Gale's Theorem to inequality (4.2), we see that if $\mathbf{A}^T\mathbf{y} = 0$ has no non-zero solution $\mathbf{y}$ then a solution $\mathbf{x}$ exists for inequality (4.2). □

Notice that this proof does not depend on the choice of ϵ. Geometrically, this implies that as long as the hypothesis of the theorem holds we can find polynomials of degree d that not only complete the knot, but leave arbitrarily large gaps at the crossings.

Using the previous theorem and basic theorems from linear algebra we can obtain a computationally useful sufficiency conditions for the existence of a solution.

Corollary 5.6. *Let k be a knot-type and let $p(t) = (f(t), g(t), 0)$ be a c- crossing polynomial projection of k. Let $\mathbf{A}$ be the degree d crossing pair matrix of $p(t)$. If rank$(\mathbf{A}) = c$, then there is a polynomial $h(t)$ of degree $r \leq d$ that completes the knot.*

5.4. **An Example.** Consider the following polynomial projection of the trefoil knot:

$$f(t) = 3t^4 - 2t^3 - 3t^2 + t4$$
$$g(t) = 5t^5 - t^4 + t^3 - 5t^2 - 5t.$$

Assuming that the first crossing is an over-crossing and computing the three crossing point pairs of this projection yields the approximate solutions

$$(u_1, v_1) = (-0.859, 0.425), (u_2, v_2) = (1.179, -0.799), (u_3, v_3) = (-0.073, 1.241).$$

We now compute the degree 5 crossing pair matrix for this projection. Note that since this is a 3-crossing alternating projection by the method with which we constructed a solution in the proof of Theorem 3.3 there must be a polynomial of degree 5 that completes this knot.

$$\mathbf{A}_5 = \begin{pmatrix} u_1^d - v_1^d & \cdots & u_1^1 - v_1^1 \\ \vdots & \ddots & \vdots \\ u_c^d - v_c^d & \cdots & u_c^1 - v_c^1 \end{pmatrix} = \begin{pmatrix} 1.284 & 0.557 & -0.711 & 0.512 & -0.482 \\ 1.978 & 0.752 & 2.149 & 1.525 & 2.604 \\ -1.314 & -1.535 & -1.912 & -2.372 & -2.943 \end{pmatrix}$$

It can be verified that the first three columns form a basis of size three for the column space of $\mathbf{A}_5$. But the three columns form the submatrix $\mathbf{A}_3$ of $\mathbf{A}_5$. So we see that $rank(\mathbf{A}_3) = 3$, and hence there is a polynomial of degree 3 or less that completes this knot. To obtain the actual solution, we simply add multiples of $f(t)$ and $g(t)$ to a degree 5 solution obtained using the method in the proof of Theorem 3.3 to eliminate the degree 4 and degree 5 terms.

6. Further Questions

While we now have a condition of sufficiency for the existence of a low-degree solution to the knot completion problem, the condition does depend on the choice of functions and value of the parameter at the crossing pairs representing the projection. Notice that the crossing pairs can be "changed" by doing a linear change of variables on the projection's parameterization without changing the projection at all. It would be interesting to see if a stronger sufficiency condition (or a necessary condition–which was what the original goal of this research was) that only depends on the number of crossings can be found using this approach.

Throughout this paper we have restricted ourselves to one projection of a knot represented by three functions of an orthogonal basis of $\mathbb{R}^3$. By relaxing the orthogonal basis restriction, we can find a basis so that the projections of a knot-type k on the planes defined by any two of the basis vectors would have c crossings. We could then potentially look at all three projections simultaneously and attempt to determine properties of all three polynomials instead of just one of them.

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