THE SPACE OF DEGREE-THREE POLYNOMIAL KNOTS

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1. Introduction

In his 1995 paper, "On The Spaces of Polynomial Knots," Victor Vassiliev claims that for any $n \geq 2$, the parameter space of degree-three polynomial embeddings $\kappa : \mathbb{R} \to \mathbb{R}^n$of the form

$$\kappa(t) = \begin{pmatrix} x_1(t) & = & t^3 & + & a_{1_2}t^2 & + & a_{1_1}, \\ x_2(t) & = & t^3 & + & a_{2_2}t^2 & + & a_{2_1}, \\ \vdots & = & \vdots & + & \vdots & + & \vdots \\ x_n(t) & = & t^3 & + & a_{n_2}t^2 & + & a_{n_1} \end{pmatrix}$$

is contractable and, in fact, star-shaped about the point

$$\kappa_0(t) = \begin{pmatrix} x_1(t) & = & t^3 & + & t, \\ x_2(t) & = & t^3 & + & t, \\ \vdots & = & \vdots & + & \vdots \\ x_n(t) & = & t^3 & + & t \end{pmatrix}$$

Vassiliev does not provide a proof of this in his paper, saying that the claim is "trivial, and should be known to the specialists." Here we provide a proof of Vassiliev's assertion.

2. Background Information

We will begin with a couple of basic definitions.

Definition 1. A vector space S over the reals is said to be **star-shaped** if there exists at least one point $\vec{s_0} \in S$ such that for any point $\vec{s} \in S$, and any real $r \in [0, 1]$ we have $\vec{s_0} + r(\vec{s} - \vec{s_0}) \in S$.

Definition 2. An n-dimensional polynomial parametrization is said to be an **embedding** if it satisfies two conditions:

(1) $X(p) = X(q) \Rightarrow p = q$
(2) $\forall\ t \in \mathbb{R}$, $X'(t) \neq 0$

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So in order for a parametrization to be an embedding, it must be one-to-one and have no zero-derivatives.

Now we look at the specific space we are working with: the space of degree-three embeddings. Any polynomial parametrization of the form

$$X = \begin{pmatrix} x_1(t) & = & t^3 & + & a_{1_2}t^2 & + & a_{1_1}, \\ x_2(t) & = & t^3 & + & a_{2_2}t^2 & + & a_{2_1}, \\ \vdots & = & \vdots & + & \vdots & + & \vdots \\ x_n(t) & = & t^3 & + & a_{n_2}t^2 & + & a_{n_1} \end{pmatrix}$$

can be rewritten as

$$X = \begin{pmatrix} x_1(t) = t^3 + t + (a_{1_2}t^2 + (a_{1_1} - 1)t), \\ x_2(t) = t^3 + t + (a_{2_2}t^2 + (a_{2_1} - 1)t), \\ \vdots \\ x_n(t) = t^3 + t + (a_{n_2}t^2 + (a_{n_1} - 1)t) \end{pmatrix}$$

For any parametrization X of this form, and any real number r, let

$$X_r = \begin{pmatrix} x_{1_r}(t) = t^3 + t + r(a_{1_2}t^2 + (a_{1_1} - 1)t), \\ x_{2_r}(t) = t^3 + t + r(a_{2_2}t^2 + (a_{2_1} - 1)t), \\ \vdots \\ x_{n_r}(t) = t^3 + t + r(a_{n_2}t^2 + (a_{n_1} - 1)t) \end{pmatrix}$$

Note that if we take the embedding

$$X_0(t) = \begin{pmatrix} x_1(t) & = & t^3 & + & t, \\ x_2(t) & = & t^3 & + & t, \\ \vdots & = & & \vdots & \\ x_n(t) & = & t^3 & + & t \end{pmatrix}$$

as the point about which our parameter space is to be star-shaped, then for any embedding X, $X_r = X_0 + r(X - X_0)$. Therefore the space will be star-shaped about the point X_0 if, given an embedding X and any real $r \in [0, 1]$, X_r is also an embedding.

3. Proving Vassiliev's assertion

Lemma 1. *If for some $X : \mathbb{R} \to \mathbb{R}^n$ there exists $t \in \mathbb{R}$ such that $X'(t) = 0$, then X_r will have a double point for all $r > 1$.*

Proof. Suppose that for some polynomial parametrization X and some real number s, $X'(s) = 0$. Then for any $i \in \{1, 2, \ldots, n\}$,

$$x_i'(s) = 3s^2 + 2a_2 s + a_1 = 0.$$

For any $r > 1$, let

$$\begin{aligned} p &= s + \left(\sqrt{3s^2+1}\right)\left(\sqrt{r-1}\right), \\ q &= s - \left(\sqrt{3s^2+1}\right)\left(\sqrt{r-1}\right). \end{aligned}$$

To show that $x_{i_r}(p) = x_{i_r}(q)$, it will be sufficient to prove that $\left(x_{i_r}(p) - x_{i_r}(q)\right) / (p - q) = 0$. Note that since $r \neq 1$, we have $p \neq q$.

$$\begin{aligned} \frac{x_{i_r}(p) - x_{i_r}(q)}{p-q} &= p^2 + pra_{2_i} + pq - r + ra_{1_i} + 1 + qra_{2_i} + q^2 \\ &= \left(s + \left(\sqrt{3s^2+1}\right)\left(\sqrt{r-1}\right)\right)^2 + \left(s + \left(\sqrt{3s^2+1}\right)\left(\sqrt{r-1}\right)\right) ra_{2_i} + \\ &\quad \left(s + \left(\sqrt{3s^2+1}\right)\left(\sqrt{r-1}\right)\right)\left(s - \left(\sqrt{3s^2+1}\right)\left(\sqrt{r-1}\right)\right) - r + ra_{1_i} + \\ &\quad \left(s - \left(\sqrt{3s^2+1}\right)\left(\sqrt{r-1}\right)\right) ra_{2_i} + \left(s - \left(\sqrt{3s^2+1}\right)\left(\sqrt{r-1}\right)\right)^2 \\ &= 2s^2 + 2(3s^2+1)(r-1) + 2sra_{2_i} + s^2 - (3s^2+1)(r-1) - r + ra_{1_i} + 1 \\ &= 3s^2 + (3s^2+1)(r-1) + 2sra_{2_i} - r + ra_{1_i} + 1 \\ &= 3s^2 + 3rs^2 - 3s^2 + r - 1 + 2sra_{2_a} - r + ra_{1_i} + 1 \\ &= 3s^2r + 2na_{2_i}r + a_{1_i}r \\ &= r(3s^2 + 2sa_{2_i} + a_{1_i}) \\ &= 0 \end{aligned}$$

Thus when $X'(s) = 0$, we have $X_r\left(s + \left(\sqrt{3s^2+1}\right)\left(\sqrt{r-1}\right)\right) = X_r\left(s - \left(\sqrt{3s^2+1}\right)\left(\sqrt{r-1}\right)\right)$ for all $r > 1$. We know that for real $r > 1$, $s + \left(\sqrt{3s^2+1}\right)\left(\sqrt{r-1}\right) \neq s - \left(\sqrt{3s^2+1}\right)\left(\sqrt{r-1}\right)$, so from this we have Lemma 1. □

Lemma 2. *If for some $X : \mathbb{R} \to \mathbb{R}^n$ there exist p and q such that $X(p) = X(q)$, and $p \neq q$, then there exists $r \in (0,1]$ such that $X'_r\left((p-q)/2\right) = 0$.*

Proof. Suppose that for some polynomial parametrization X and some real p and q such that $p \neq q$, $X(p) = X(q)$. It follows that for any x_i,

$$\frac{x_i(p) - x_i(q)}{p-q} = 0$$

This equation expands out to

$$p^2 + pa_{2_i} + pq + a_{1_i} + qa_{2_i} + q^2 = 0.$$

We collect and rearrange terms to get

$$(p+q)a_{2_i} + a_{1_i} = -(p^2 + pq + q^2).$$

We can then factor a 2 out of the first term and subtract 1 from both sides to get

$$2a_{2_i}\left(\frac{p+q}{2}\right)+a_{1_i}-1=-(p^2+pq+p^2+1)$$

We can then multiply both sides by

$$\frac{3\left(\frac{p+q}{2}\right)^2+1}{p^2+pq+q^2+1}.$$

This gives us

$$\frac{3(p+q)^2+4}{4(p^2+pq+q^2+1)}\left(2a_{2_i}\left(\frac{p+q}{2}\right)+a_{1_i}-1\right)=-\left(3\left(\frac{(p+q)}{2}\right)^2+1\right)$$

We can now rearrange terms to get

$$3\left(\frac{p+q}{2}\right)^2+1+\frac{3(p+q)^2+4}{4(p^2+pq+q^2+1)}\left(2a_{2_i}\left(\frac{p+q}{2}\right)+a_{1_i}-1\right)=0$$

The expression on the left is, in fact,

$$x'_{i_r}\left(\frac{p+q}{2}\right),$$

for

$$r=\frac{3(p+q)^2+4}{4(p^2+pq+q^2+1)}.$$

So all that remains is to prove that $0<r\leq 1$.

First we will prove that $0<r$. We know that

$$3(p+q)^2+4>0$$

So now we need to prove that the denominator is positive. We know

$$(p+q)^2\geq 0$$

It follows that

$$p^2+q^2+2pq\geq 0$$

If pq is positive, then the denominator is some non-negative number plus one, and we're done. If pq is negative, then it follows that

$$|p^2+q^2|\geq|2pq|$$

Finally it follows that

$$|p^2+q^2|\geq|pq|$$

So we have

$$4(p^2+pq+q^2+1)>0$$

and so r must be greater than zero.

Similarly, we can prove that $r \leq 1$. We know that

$$(p-q)^2 \geq 0.$$

We expand and rearrange terms to get

$$2pq \leq p^2 + q^2.$$

We then add $3p^2 + 4pq + 3q^2$ to both sides to get

$$3p^2 + 6pq + 3q^2 + 4 \leq 4p^2 + 4pq + 4q^2 + 4.$$

We divide out by the right side to get

$$\frac{3p^2 + 6pq + 3q^2 + 4}{4p^2 + 4pq + 4q^2 + 4} \leq 1.$$

We then do some factoring to get

$$\frac{3(p+q)^2 + 4}{4(p^2 + pq + q^2 + 1)} \leq 1.$$

Thus if X has a double point, there exists $r \leq 1$ such that X_r has a zero-derivitive at $(p+q)/2$. So we have Lemma 2. □

The main result follows easily from these two lemmas.

Theorem 1. *The parameter space of degree-three polynomial embeddings $\kappa : \mathbb{R} \to \mathbb{R}^n$ of the form*

$$\kappa(t) = \begin{pmatrix} x_1(t) & = & t^3 & + & a_{1_2}t^2 & + & a_{1_1}, \\ x_2(t) & = & t^3 & + & a_{2_2}t^2 & + & a_{2_1}, \\ \vdots & = & \vdots & + & \vdots & + & \vdots \\ x_n(t) & = & t^3 & + & a_{n_2}t^2 & + & a_{n_1} \end{pmatrix}$$

is star-shaped about the point

$$\kappa_0(t) = \begin{pmatrix} x_1(t) & = & t^3 & + & t, \\ x_2(t) & = & t^3 & + & t, \\ \vdots & = & \vdots & + & \vdots \\ x_n(t) & = & t^3 & + & t \end{pmatrix}$$

Proof. Suppose X is an embedding from $\mathbb{R}$ to $\mathbb{R}^n$. Suppose X_r is not an embedding for some $0 \leq r \leq 1$. Then either X_r has a zero-derivative or X_r is not one-to-one. If X_r has a zero-derivative, then X has a double point by Lemma 1. This would present a contradiction. If X_r is not one-to-one, then by Lemma 2, there exists $\hat{r} \leq r$ such that $X_{\hat{r}}$ has a zero-derivative. The by Lemma 1, X_r has a double point, and is therefore not an embedding. This, too, presents a contradiction. Therefore, if X is an embedding, then X_r is also an embedding for all

$0 \leq r \leq 1$. This proves that our space is star-shaped about κ_0, and therefore Vassiliev's assertion is true. $\square$