Counting Nodes on Rational Plane Curves

Charles Siegel

April 29, 2007

Abstract

In this paper, we consider polynomial parametrized curves in the affine plane k^2 over an algebraically closed field k. Such curves are given by $\kappa : k \to k^2 : t \mapsto (x(t), y(t))$, and may or may not contain singular points. The problem of how many singular points there are is of specific importance to the theory of polynomial knots, as it gives a bound on the degrees necessary to achieve a parametrization of a knot with a specified number of crossings. We will state a more general conjecture and supply a proof of it in a special case.

1 Introduction

Our goal is to relate the number of singularities in a map $\kappa : k \to k^n$ to the image of the induced map on rings $\kappa^* : k[x_1, \dots, x_n] \to k[t]$. Ideally, we would like to prove the following conjecture:

Conjecture 1. *Let k be an algebraically closed field, and let $\kappa : k \to k^n$ be a polynomial map. Then the number of singular points of κ is the dimension of the vector space $k[t]/\operatorname{Im}\kappa^*$*

Unfortunately, this is rather difficult. There are, however, cases where progress can be made. Specifically, the case where κ is an embedding is a classical result, which this conjecture should be considered as generalization of. Also, we will show that the conjecture holds for rational plane curves with $\deg x = d, \deg y = d - 1$ and only nodes for singularities.

2 Embeddings

We begin with a classical theorem about algebraic varieties which contains, as a special case, the statement that a map $\kappa : k \to k^n$ is an embedding if and only if $\kappa^* : k[x_1, \dots, x_n] \to k[t]$ is surjective, which proves the conjecture in the case of embeddings.

Theorem 1. *Let X, Y be affine varieties over an algebraically closed field k. Let $\varphi : X \to Y$ be a morphism of affine varieties. Then this morphism is an embedding if and only if $\varphi^* : k[Y] \to k[X]$ is surjective.*

Proof. Let X, Y be affine varieties. If $X \subseteq Y$ is a closed subvariety, that is, $\phi : X \to Y$ is an embedding, then $k[X] \simeq k[Y]/I(X)$, so φ^* is surjective.

Conversely, if φ^* is surjective, then let $I = \ker \varphi^* \subseteq k[Y]$, so $X \simeq V(I)$ and $V(I)$ is a closed subvariety of Y. Thus, φ is an embedding. $\square$

3 Plane Curves

Before moving on to plane curves proper, we will prove the case of maps $\kappa : k \to k$.

Proposition 1. *Let k be an algebraically closed field. Let $\kappa : k \to k$ be a polynomial map. Then the number of singular points of κ, if finite, is the dimension of the vector space $k[t]/\operatorname{Im}\kappa^*$, and if one of the two is infinite, the other is as well.*

Proof. For this case, we have that $\kappa(t) = c_n t^n + \ldots + c_0$ for some $n \in \mathbb{N}$ and $c_n \neq 0$.

We separate into two cases. If κ is an embedding, then κ^* is surjective and we are done.

If $\deg \kappa(t) = d$, then for each $c \in k$, there are d solutions, counted with multiplicity, to the equation $\kappa(t) = c$. If there is a multiple root, then $\kappa'(t) = 0$ there, else $\kappa(t)$ must intersect itself at c unless $n = 1$, in which case $\kappa(t)$ is an embedding.

So, as κ is not an embedding, $d \geq 2$. We take the basis of $k[t]$ to be the monomials, and note that every complement of a subspace has the same dimension, and has the same dimension as the quotient. The complement of $\operatorname{Im}\kappa^*$ has basis $\{1, t, \ldots, t^{n-1}, t^{n+1}, \ldots, t^{2n-1}, t^{2n+1}, \ldots\}$. That is, it is infinite dimensional, which completes the proof. $\square$

Before discussing the case for plane curves, we will review some definitions and standard results that will be necessary.

Definition 1 (Node)**.** *A curve $\kappa : k \to k^2$ has a node if $t_1, t_2 \in k$ such that $\kappa(t_1) = \kappa(t_2)$ and $\kappa'(t_1) \neq \kappa'(t_2)$.*

Definition 2 (Degree of a Curve)**.** *A plane curve defined by $f(x, y) = 0$ is said to have degree $n = \deg f$ when f is an irreducible polynomial in two variables over k.*

We will need the following standard results:

Lemma 1. *Let $C = V(f)$ be a plane curve of genus g and degree d with only nodes for singularities. Then $g = \frac{(d-1)(d-2)}{2} - \delta$, where δ is the number of nodes of C.*

This result can be found in [Sha94] or scattered throughout the exercises of [Har77], and we will use it without proof.

With the above set up completed, we can now prove a portion of the conjecture.

Theorem 2. *Let k be an algebraically closed field, and let $\kappa : k \to k^2$ be a polynomial map with $\deg x(t) = d$ and $\deg y(t) = d-1$ and only nodes for singular points. Then the number of nodes of κ is the dimension of the vector space $k[t]/\operatorname{Im}\kappa^*$. Additionally, this number is equal to $\frac{(d-1)(d-2)}{2}$.*

Proof. First, we look at the formula for the lemma, namely, $g + \delta = \frac{(n-1)(n-2)}{2}$ where n is the degree of the curve. We now note that the genus of the affine line is 0, and that the genus is invariant so $\operatorname{Im}\kappa$ has genus 0. Thus, $\delta = \frac{(n-1)(n-2)}{2}$.

We now approach from the other direction, and attempt to determine the dimension of the specified vector space. We intend to show that it must be $\frac{(d-1)(d-2)}{2}$. We take $x(t) = t^d + \ldots + a_0$ and $y(t) = t^{d-1} + \ldots + b_0$. We take a basis $\{1, t, t^2, \ldots\}$ for $k[t]$ and, as we are attempting to determine the dimension of the quotient space $k[t]/\operatorname{Im}\kappa^*$, we note that it has the same dimension as any complement of $\operatorname{Im}\kappa^*$, and thus, we can merely look for a set of monomials that, if adjoined to $\operatorname{Im}\kappa^*$, would give us $k[t]$.

If j is in the numerical monoid generated by $d, d-1$, then t^j is linearly dependent on $\{1, t, \ldots, t^{j-1}\}$, as then there exist i, ℓ such that t^j is the lead term of $x^i y^\ell$, and so $t^j - x^i y^\ell = \sum_{i=0}^{j-1} a_i t^i$. We desire to show that the set $\{t^i : i$ is not in the monoid generated by $d, d-1\}$ is a linearly independent set.

As $d, d-1$ are relatively prime, we know that there are only finitely many elements of this set. Let $i_1, \ldots, i_\alpha$ be these i. Eventually, we shall see that $\alpha = \frac{(d-1)(d-2)}{2}$. We now let $r_1, \ldots, r_\alpha \in k$ such that $r_1 t^{i_1} + \ldots + r_\alpha t^{i_\alpha} = 0$ in $k[t]/\operatorname{Im}\kappa^*$. That is, $r_1 t^{i_1} + \ldots + r_\alpha t^{i_\alpha} \in \operatorname{Im}\kappa^*$. Any element of $\operatorname{Im}\kappa^*$ can be written as $\sum_\beta c_\beta t^\beta$ with at least one nonzero c_β with β in the monoid generated by d and $d-1$. So, we have $r_1 t^{i_1} + \ldots + r_\alpha t^{i_\alpha} = \sum_\beta c_\beta t^\beta$ for some $r_i, c_\beta \in k$ inside the vector space $k[t]$. From this it follows that all of the r_i, c_β must be zero, as the t^i, $i \in \mathbb{N}$ form a basis for $k[t]$.

So all that remains is counting the number of elements not in the numerical monoid generated by $d, d-1$. Let $a, b, c \in \mathbb{N}$, $b < d$, $c < d-1$, then any element of the monoid can be written in the form $ad(d-1) + bd + c(d-1)$. We must next choose a number larger than the Frobenius number, say, $d(d-1)$. Then finding all a, b, c such that $ad(d-1) + bd + c(d-1) < d(d-1)$ holds counts the number of elements of the monoid less than $d(d-1)$. By necessity, this inequality forces $a = 0$, and so we have $bd + c(d-1) < d(d-1)$, $b < d$, $c < d-1$ as a system to solve. We note that as d and $d-1$ are relatively prime, $bd + c(d-1)$ gives no number twice.

This system of inequalities gives us a triangular region in the integer lattice and we can apply Pick's Theorem [Var85]. This theorem states that for P a lattice polygon, $A(P)$ the area of P, $I(P)$ the number of points on the interior and $B(P)$ the number of points on the boundary, the equality $A(P) = I(P) + B(P)/2 - 1$ holds. As P is a triangle with sides of length $d-1$ and $d-2$, we have that $\frac{(d-1)(d-2)}{2} = I(P) + B(P)/2 - 1$, with the right hand side being equal to the number of points in the monoid. And so, there are $\frac{(d-1)(d-2)}{2}$ elements of the monoid less than $d(d-1)$.

This expression is actually half of the Frobenius number plus one, and so this tells us that only half of the numbers less than the Frobenius number can be in the monoid. Thus, half must be out of it, and so $\frac{(d-1)(d-2)}{2}$ is the number of elements not in the monoid.

Thus, the number of monomials that are not the lead term of anything in the image must be $\frac{(d-1)(d-2)}{2}$, and so, from the above statement that the number of such monomials is the dimension of $k[t]/\operatorname{Im}\kappa^*$, we have the theorem.

The only thing that remains is to show that $d = n$, that is, that the degree of the parametrization and the degree of the curve are equal. The degree is equal to the number of points of intersection (with multiplicity) of the curve with a line in the projective plane. We get the equality by taking the line $x = 0$ and seeing that the curve will satisfy that equation precisely d times, so $n = d$. $\square$

This gives an alternate proof of Lemma 4 in [DO], as if δ is the number of nodes, then $\leq \delta$ of them are real, and so the crossing number of a polynomial knot $c \leq \delta = \frac{(d-1)(d-2)}{2}$. More importantly, however, it suggests that Conjecture 1 may be true, with the appropriate notion of the multiplicity of a singular point.

4 A Motivating Example

As our first example, we have the Trefoil knot in three space. It is given by the parametrization

$$\begin{aligned} x(t) &= t^3 - 3t \\ y(t) &= t^4 - 4t^2 \\ z(t) &= t^5 - 10t \end{aligned}$$

Through the methods of Gröbner bases, we find that $t = -\frac{1}{4}yx - \frac{5}{4}x^2z - \frac{1}{4}zy^2 + \frac{1}{4}xz^2 - yz + \frac{11}{2}x - \frac{7}{4}z$. And so, we see that in three dimensions the polynomials themselves do in fact matter, as we can get a degree 1 polynomial out of polynomials of degree $3, 4$ and 5, and 1 is certainly not in the monoid generated by $\{3, 4, 5\}$.

Though the three dimensional trefoil is an embedding, the three projections onto the xy, yz and xz planes are not. First we look at the projection into the xy plane. We can check easily that there are exactly three nodes. Similarly, the monoid generated by $\{3, 4\}$ only misses $1, 2$ and 5, and there is no way to get polynomials of these degrees in $\operatorname{Im}\kappa^*$, thus, the two methods of counting match up. Similarly the yz, the monoid on $\{4, 5\}$ misses 6 numbers, $\{1, 2, 3, 6, 7, 11\}$, and we find that there are three real singularities and three complex ones, each given by two real or two nonreal roots of the resultant.

5 Future Work

In addition to work on proving the higher dimensional case and the case for plane curves not satisfying all of the hypotheses of the theorem, there are two possible ways to attempt to generalize the conjecture even further. The one more applicable to the theory of polynomial knots is to try to find a way to expand the conjecture to non-algebraically closed fields, hoping to find results over $\mathbb{R}$ to give better bounds on the crossing numbers that can be achieved by polynomials of a given degree. The other way to generalize is to morphisms of affine varieties whose domain is not $\mathbb{A}^1$, but rather some more arbitrary variety, as this can be seen as a measure of how much the morphism differs from an embedding.

References

[DO] Alan Durfee and Donal O'Shea. Polynomial knots.

[Eis95] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[GP02] Gert-Martin Greuel and Gerhard Pfister. *A* **Singular** *introduction to commutative algebra*. Springer-Verlag, Berlin, 2002. With contributions by Olaf Bachmann, Christoph Lossen and Hans Schönemann, With 1 CD-ROM (Windows, Macintosh, and UNIX).

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Sha94] Igor R. Shafarevich. *Basic algebraic geometry. 1*. Springer-Verlag, Berlin, second edition, 1994. Varieties in projective space, Translated from the 1988 Russian edition and with notes by Miles Reid.

[Var85] Dale E. Varberg. Pick's theorem revisited. *Amer. Math. Monthly*, 92(8):584–587, 1985.