UNOBSERVED HILBERT MODULAR DEFORMATION PROBLEMS

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Abstract. Let \( \rho_{f,\lambda} \) be an \( \ell \)-adic Galois representation associated to a Hilbert newform \( f \). Consider its semisimple mod \( \ell \) reduction \( \bar{\rho}_{f,\lambda} \). This paper discusses how, under certain conditions on \( f \), the universal ring for deformations of \( \bar{\rho}_{f,\lambda} \) with fixed determinant is unobstructed for almost all primes. We follow the approach of Weston, who carried out a similar program for classical modular forms in 2004. As such, the problem essentially comes down to verifying that various local invariants vanish at all places dividing \( \ell \) or the level of the newform. We conclude with an explicit example illustrating how one can in principle find a lower bound on \( \ell \) such that the universal ring for deformations of \( \bar{\rho}_{f,\lambda} \) with fixed determinant is unobstructed for all \( \lambda \) over \( \ell \).

1. Introduction

Let \( f \) be a newform of level \( N \) and weight \( k \geq 2 \). Let \( K_f \) be the number field obtained from \( f \) by adjoining its Hecke eigenvalues to \( \mathbb{Q} \). For each prime \( \lambda \) in \( K_f \), Deligne constructed a semisimple mod \( \ell \) representation \( \bar{\rho}_{f,\lambda} \). In [13], Mazur conjectured that the universal deformation ring of this residual representation \( \bar{\rho}_{f,\lambda} \) is unobstructed for almost all \( \lambda \). Weston [16] gave a positive answer to Mazur’s question in 2004 assuming that \( k \geq 3 \). He was also able to obtain some results for weight two modular forms, showing that Mazur’s conjecture holds on a set of primes of density one. We show that Weston’s methodology and results essentially carry over to the Hilbert modular form setting with a few minor adjustments.

More specifically, let \( F \) be a totally real extension of \( \mathbb{Q} \) of degree \( d > 1 \) and let \( f \) be a Hilbert newform on \( F \) of level \( n \subset \mathcal{O}_F \) and weight \( k = (k_{\tau_1}, \ldots, k_{\tau_d}) \). Here the \( \tau_i \) denote the embeddings of \( F \) into \( \mathbb{R} \). We assume that \( k_{\tau_i} \geq 2 \) for all \( i \) and that they satisfy the parity condition \( k_{\tau_1} \equiv \cdots \equiv k_{\tau_d} \mod 2 \). As in the previous paragraph, let \( K_f \) be the number field generated over \( \mathbb{Q} \) by the Hecke eigenvalues of \( f \) and let \( \mathcal{O}_{K_f} \), its ring of integers. For each prime \( \lambda \) of \( K_f \), let

\[
\bar{\rho}_{f,\lambda}: G_{F,S} \to \text{GL}_2(k_{f,\lambda})
\]

be the semisimple mod \( \ell \) Galois representation attached to \( f \) by Carayol and Taylor. Here \( k_{f,\lambda} = \mathcal{O}_{K_f}/\lambda \) and \( G_{F,S} = \text{Gal}(F_S/F) \), where \( F_S \) is the maximal algebraic extension of \( F \), unramified outside of a finite set of places \( S = \{v|n\ell \} \cup \{v|\infty \} \).

Let \( D_{\bar{\rho}_{f,\lambda}}^{\text{det}=\delta} \) denote the functor that associates to a coefficient ring \( R \) the set of all deformations of \( \bar{\rho}_{f,\lambda} \) to \( R \) with fixed determinant (see section 2 for precise definitions regarding deformation theory). Note that \( \bar{\rho}_{f,\lambda} \) is absolutely irreducible for almost all \( \lambda \) [4, Proposition 3.1]. For such \( \lambda \), the functor \( D_{\bar{\rho}_{f,\lambda}}^{\text{det}=\delta} \) is representable by the
universal deformation ring $R_{f,\lambda}$ for deformations with fixed determinant. Then our main theorem is the following.

**Theorem 1.1.** Suppose that $f$ has no CM, is not a twist of a base change of a Hilbert newform on $E \subseteq F$, and $k_i \geq 3$ for all $i$. Then $R_{f,\lambda}$ is unobstructed for almost all $\lambda$.

**Remark 1.2.** Weston [16] did not have this additional condition of deformations with fixed determinant, but in general there are obstructions that come from lifting the determinant, so there is no way around this. See, for example, [14, Theorem 10.7.3] for details about calculating $\dim_{\mathbb{F}_p} H^2(G_{F,S}, \mathbb{Z}/p\mathbb{Z})$.

**Remark 1.3.** The hypotheses that $f$ has no CM and is not a twist of a base change come from ensuring that certain Selmer groups vanish for almost all $\lambda$ (see Proposition 2.4, [5, Theorem B(i)] and [6, Theorem 2.1]). It is an open problem as to whether or not these hypotheses can be relaxed.

The strategy for proving Theorem 1.1 is to use a generalization of a criterion for unobstructedness (Proposition 2.4) due to Weston [16]. Using this proposition and results of Dimitrov ([5] and [6]), the proof is reduced to checking that for all $v \in S$, the local cohomology groups $H^0(G_v, \bar{\varepsilon} \otimes \text{ad}^0 \bar{\rho}_{f,\lambda}) = 0$ for almost all $\lambda$. Here $\varepsilon$ is the $\ell$-adic cyclotomic character, $G_v$ is a decomposition group at a place $v$, and $\text{ad}^0 \rho$ denotes the restriction of the adjoint representation of $\rho$ to the trace-zero matrices. Section 3 addresses those $v \in S$ such that $v \not| \ell$, while section 4 shows that for almost all $\lambda$, this vanishing cohomology condition holds for $v|\ell$. We also give a proof of Theorem 1.1 in section 4. We conclude in Section 5 with an explicit example of determining a lower bound on $\ell$ such that $R_{f,\lambda}$ is unobstructed for all $\lambda$ over $\ell$. Here $f$ is the unique level one newform on $\mathbb{Q}(\sqrt{5})$ of weight $(4,8)$.

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**Notation.** For a field $F$, denote its absolute Galois group by $G_F$. As above, we let $G_v$ denote a decomposition group at a place $v$ of $F$ and fix embeddings $G_v \hookrightarrow G_F$. Let $F_v$ denote the $v$-adic completion of $F$. We use the phrase “almost all” as a substitute for “all but finitely many.”

2. Review of Galois deformation theory

We briefly recall the theory of deformations of mod $\ell$ Galois representations in the sense of Mazur. For a more thorough introduction see [2] or [11].
Let $F$ be a number field and let $S$ be a finite set of places of $F$. Let $k$ be a finite field of characteristic $\ell$ and denote the Witt vectors of $k$ by $W(k)$. Consider an absolutely irreducible continuous representation 

$$\bar{\rho} : G_{F,S} \to \GL_n(k).$$

Also consider the category $\mathcal{C}$ of complete local noetherian rings $R$ with residue field $k$. Morphisms in this category are local homomorphisms that induce the identity on $k$. A lift of $\bar{\rho}$ to $R$ is a continuous representation $\rho : G_{F,S} \to \GL_n(R)$ making the following diagram commute:

$$
\begin{array}{c}
G_{F,S} \\
\downarrow \rho \\
\GL_n(R) \\
\downarrow \rho \\
\GL_n(k)
\end{array}
$$

where the homomorphism $\GL_n(R) \to \GL_n(k)$ is the map induced by the reduction homomorphism $R \to k$. We say that two lifts $\rho$ and $\rho'$ of $\bar{\rho}$ to $R$ are strictly equivalent if $\gamma \rho \gamma^{-1} = \rho'$ for some $\gamma \in \ker(\GL_n(R) \to \GL_n(k))$.

**Definition 2.1.** A deformation of $\bar{\rho}$ to $R$ is a strict equivalence class of lifts of $\bar{\rho}$ to $R$.

Consider the functor $D_{\bar{\rho}} : \mathcal{C} \to \text{SETS}$ given by

$$D_{\bar{\rho}}(R) = \{\text{deformations of } \bar{\rho} \text{ to } R\}.$$ 

Call such a functor a deformation problem.

**Theorem 2.2** (Mazur). If $\bar{\rho}$ is absolutely irreducible then $D_{\bar{\rho}} = \text{Hom}(R_{\bar{\rho}}, -)$ and 

$$R_{\bar{\rho}} \cong W(k)[[x_1, \ldots, x_{d_i}]]/I.$$ 

Here $d_i = \dim_k H^1(G_{F,S}, \text{ad } \bar{\rho})$ and $I$ is generated by at most $d_2$ elements.

**Definition 2.3.** The deformation problem $D_{\bar{\rho}}$ is unobstructed if $d_2 = 0$.

We can also consider subfunctors of $D_{\bar{\rho}}$ where we ask our deformations to satisfy certain prescribed properties. For example, we can ask for deformations with fixed determinant. By this we mean that $\det \rho$ is the composition of the canonical homomorphism $W(k) \to R$ (making $R$ a $W(k)$-algebra) with a fixed continuous character $\delta : G_{F,S} \to W(k)$. When this occurs, we say that a deformation $\rho$ has det $= \delta$. Denote by $D_{\bar{\rho}}^{\det = \delta}$ the subfunctor given by

$$D_{\bar{\rho}}^{\det = \delta} = \{\text{deformations of } \bar{\rho} \text{ to } R \text{ with det } = \delta\}.$$ 

Note that for the deformation problem $D_{\bar{\rho}}^{\det = \delta}$, an analogue to Theorem 2.2 holds where we replace $\text{ad } \bar{\rho}$ by $\text{ad}^0 \bar{\rho}$ in the statement of the theorem.

We now specialize to two-dimensional residual representations $\bar{\rho} : G_{F,S} \to \GL_2(k)$. Let $K$ be a finite extension of $\mathbb{Q}_\ell$ and let $\mathcal{O}$ be its ring of integers. Assume that we have a (fixed) continuous representation 

$$\rho : G_{F,S} \to \GL_2(\mathcal{O})$$

lifting $\bar{\rho}$. Set $V_\rho = K^3$ and $A_\rho = (K/\mathcal{O})^3$. Give $V_\rho$ and $A_\rho$ a $G_F$-action via $\text{ad}^0 \rho$. Let $V_\rho(1)$ denote the Tate-twist of $V_\rho$. Finally, define the Selmer groups
Proposition 2.4. Suppose

1. \( H^0(G_v, \bar{\varepsilon} \otimes \text{ad}^0 \bar{\rho}) = 0 \) for all \( v \in S \setminus \{v|\infty\} \),
2. \( H^1_F(G_F, V_{\rho}(1)) = 0 \),
3. \( H^1_F(G_F, A_{\rho}) = 0 \).

Then \( H^2(F_S, \text{ad}^0 \bar{\rho}) = 0 \). That is, \( D^\text{det}=\delta \bar{\rho} \) is unobstructed.

Proof. The argument follows mutatis mutandis as in the proof of Proposition 2.2 in [16].

Thus the strategy for proving Theorem 1.1 is clear. For \( \rho = \rho_f,\lambda \), we need to check that the hypotheses of Proposition 2.4 hold for almost all primes \( \lambda \) of \( K_f \).

3. Local invariants for \( \ell \neq p \)

Let \( F \) be a number field and let \( v \) be a prime in \( F \) over a rational \( p \in \mathbb{Z} \). In this section, we show that the local invariants \( H^0(G_v, \bar{\varepsilon} \otimes \text{ad}^0 \bar{\rho}) \) are zero for almost all \( \lambda \) not dividing \( p \). We separate the proof into two cases based on the local Langlands correspondence for \( \text{GL}_2(F_v) \).

Let \( K \) be any number field with ring of integers \( \mathcal{O} \). For all primes \( \lambda \) of \( \mathcal{O} \) not dividing \( p = v \cap \mathbb{Z} \), fix an isomorphism \( \iota_\lambda : \mathbb{C} \to \overline{\mathbb{K}}_\lambda \) extending the inclusion \( \mathcal{O} \to \mathcal{O}_\lambda \).

Let \( \mathcal{L} \) be a finite extension of \( \mathbb{Q}_p \). We say that a continuous character \( \chi : \mathcal{L} \to \mathbb{C} \) is of Galois-type with respect to \( \iota_\lambda \) if the character \( \iota_\lambda \circ \chi \) extends to a continuous character \( \chi_\lambda : G_L \to \overline{\mathbb{K}}_\lambda \) via the dense embedding \( L^\times \hookrightarrow G_L^\text{ab} \) of local class field theory. Call \( \chi \) arithmetic if \( \chi(L^\times) \subset \mathbb{Q}_p^\times \).

Let \( \pi \) be an irreducible admissible complex representation of \( \text{GL}_2(F_v) \). Call \( \pi \) arithmetic if it satisfies one of the following conditions:

- \( \pi \) is a subquotient of an induced representation \( \pi(\chi_1, \chi_2) \) where the \( \chi_i : F_v^\times \to \mathbb{C}^\times \) are arithmetic characters (i.e., \( \pi \) is principal series or special, coming from arithmetic characters),
- \( \pi \) is the base change of an arithmetic quadratic character \( \chi : L^\times \to \mathbb{C}^\times \) where \( L/F_v \) is a quadratic extension (i.e., \( \pi \) is supercuspidal and comes from the base change of an arithmetic character),
- \( \pi \) is extraordinary.

Lemma 3.1. Let \( \pi \) be an arithmetic irreducible admissible complex representation of \( \text{GL}_2(F_v) \). Let \( \{\rho_\lambda : G_v \to \text{GL}_2(K_\lambda)\} \) be a family of continuous representations for \( \lambda \) not dividing \( p \) such that \( \pi \) and \( \rho_\lambda \) are in Langlands correspondence with respect to \( \iota_\lambda \) for all \( \lambda \). If \( \pi \) is principal series or supercuspidal then

\[ H^0(G_v, \bar{\varepsilon} \otimes \text{ad}^0 \bar{\rho}_\lambda) = 0 \]

for almost all \( \lambda \).

Proof. This follows precisely as in [16, Proposition 3.2], so we do not repeat the argument here.

Corollary 3.2. We have \( H^0(G_v, \bar{\varepsilon} \otimes \text{ad}^0 \bar{\rho}_\lambda) = 0 \) for almost all \( \lambda \).
Proof. This is clear from Proposition 3.1 since
\[ \dim_{\mathbb{F}_p} H^0(G_v, \xi \otimes \text{ad}^0 \bar{\rho}_\lambda) \leq \dim_{\mathbb{F}_p} H^0(G_v, \xi \otimes \text{ad}^0 \bar{\rho}_\lambda^{ss}). \]
\[ \square \]

Note that for \( \rho_\lambda \) as in Lemma 3.1,
\[ \dim_{\mathbb{F}_p} H^0(G_v, \xi \otimes \text{ad}^0 \bar{\rho}_\lambda^{ss}) = 1 \]
for almost all \( \lambda \) when \( \pi \) is either one-dimensional or special. Although the stronger vanishing result fails when \( \pi \) is either one-dimensional or special, we can show the sufficient (and desired) vanishing of \( H^0(G_v, \xi \otimes \text{ad}^0 \bar{\rho}_\lambda) \) for almost all \( \lambda \) by using a level-lowering argument.

Now suppose that \( \pi \) is special. That is, suppose that it is the infinite dimensional quotient of \( \pi(\chi \cdot | \cdot, \chi) \) for some arithmetic character \( \chi : F^\times_v \to \mathbb{C}^\times \). Assume that \( \rho_\lambda : G_{F,S} \to \text{GL}_2(K_\lambda) \) is a \( \lambda \)-adic Galois representation such that \( \rho_\lambda|_{G_v} \) is in Langlands correspondence with \( \pi \). Then
\[ \rho_\lambda|_{G_v} \cong \left( \begin{array}{cc} \xi \chi_\lambda & * \\ 0 & \chi_\lambda \end{array} \right) \]
where \( * \) is nonzero and ramified. Here we use the fact that the norm character corresponds to the compatible system of \( G_F \)-characters \( \{ \varepsilon_\lambda := \varepsilon \} \). Set \( k_\lambda = \mathcal{O}_F/v \).

Lemma 3.3. Suppose that \( q^2 \neq 1 \mod \lambda \) where \( q = \#(\mathcal{O}_F/v) \). Then
\[ \bar{\rho}_\lambda|_{G_v} \otimes \bar{k}_\lambda \cong \left( \begin{array}{cc} \bar{\xi} \chi_\lambda & * \\ 0 & \bar{\chi}_\lambda \end{array} \right) \]
where \( * \) may be zero.

Proof. It is clear that the semi-simplification of \( \bar{\rho}_\lambda|_{G_v} \otimes \bar{k}_\lambda \) has the form \( \bar{\xi} \chi_\lambda \oplus \bar{\chi}_\lambda \), so it suffices to show that \( \bar{\rho}_\lambda|_{G_v} \otimes \bar{k}_\lambda \) is not of the form
\[ \left( \begin{array}{cc} \bar{\chi}_\lambda & \nu \\ 0 & \bar{\xi} \chi_\lambda \end{array} \right) \]
where \( \nu \) is nontrivial. It is straightforward to check that \( \bar{\xi}^{-1} \bar{\chi}^{-1} \nu \) is a 1-cocycle in \( H^1(G_v, \bar{k}_{f,\lambda}(-1)) \). Consider the inflation-restriction exact sequence
\[ H^1(G_{F,v}, \bar{k}_{\lambda}(-1)^{I_v}) \to H^1(G_v, \bar{k}_{\lambda}(-1)) \to H^1(I_v, \bar{k}_{\lambda}(-1))^{G_{F,v}} \]
where \( I_v \subset G_v \) is the inertia subgroup. An easy calculation shows that in general
\[ H^1(G_{F,v}, \bar{k}_{\lambda}(-1)^{I_v}) = 0 \] and that \( H^1(I_v, \bar{k}_{\lambda}(-1))^{G_{F,v}} = 0 \) when \( \lambda \nmid q^2 - 1 \). \( \square \)

Lemma 3.4. Suppose \( 2(q^2 - 1) \neq 0 \mod \lambda \). Then \( H^0(G_v, \text{ad}^0 \bar{\rho}_\lambda) \neq 0 \) if and only if \( \text{ad}^0 \bar{\rho}_{f,\lambda} \) is semi-simple.

Proof. This is a straightforward matrix calculation using Lemma 3.3. For example, choose the basis \( \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & 0 \\ 1 & 1 \end{array} \right) \) of \( \text{End}(V) \) where \( V \) is the 3-dimensional \( k_{f,\lambda} \)-vector space endowed with a \( G_{F,S} \) action by \( \text{ad}^0 \bar{\rho}_{f,\lambda} \). By Lemma 3.3,
\[ \bar{\rho}_\lambda \otimes \bar{k}_\lambda \cong \left( \begin{array}{cc} \bar{\xi} \chi_\lambda & \nu \\ 0 & \bar{\chi}_\lambda \end{array} \right) , \]
Thus it is clear that if \( \nu = 0 \) then \( H^0(G_v, \text{ad}^0 \bar{\rho}_\lambda) \neq 0 \). Conversely, if \( \nu \) is nonzero then using the fact that it is ramified while \( \bar{e} \) is not, one checks that there are no Galois invariants.

Let \( \rho_{f,\lambda}: G_{F,S} \to \text{GL}_2(K_{f,\lambda}) \) be the Galois representation attached to a Hilbert newform \( f \) of level \( n \), weight \( k \) and character \( \psi \) by Carayol [3] and Taylor [15]. Write \( \psi = \psi_f \cdot \bar{\varepsilon} \bar{n}_0^{-2} \) where \( \psi_f \) is a character of finite order and \( |\cdot| \) is the norm character. Note that \( \det \rho_{f,\lambda} = \psi_f^{-1} \bar{e}^{-1-k_0} \) where here \( \psi_f \) also denotes by abuse of notation the corresponding Galois character. (We find it more convenient to work with this cohomological normalization rather than the usual normalization.)

Let \( \pi \) be the automorphic representation corresponding to \( f \). Write \( \pi = \otimes' \pi_v \) for the decomposition of \( \pi \) into its irreducible admissible complex representations of \( \pi_v \) into \( \text{GL}_2(F_v) \). Fixing isomorphisms \( \iota_\lambda: \mathbb{C} \to K_{f,\lambda} \), Carayol [3, Théorème B] showed that each \( \pi_v \) is arithmetic and is in Langlands correspondence with \( \rho_{f,\lambda}|_{G_v} \) for \( \lambda \) not dividing \( p \).

**Remark 3.5.** The irreducible admissible representation \( \pi_v \) must be infinite dimensional so nothing is lost by assuming that \( \pi_v \) is special (as opposed to one-dimensional) in what follows.

**Proposition 3.6.** If \( \pi_v \) is special then

\[
H^0(G_v, \bar{e} \otimes \text{ad}^0 \bar{\rho}_{f,\lambda}) = 0
\]

for almost all \( \lambda \).

**Proof.** Note that \( \pi_v \) has central character \( \chi^2 \cdot |\cdot| \) where \( \chi \) is an arithmetic character giving rise to \( \pi_v \). By the local Langlands correspondence, this yields the equality \( \chi^2 = \psi_f^{-1} \cdot |\cdot|^{-k_0} \) where \( \psi_f \) is the \( v \)-component of \( \psi_f \). Set \( \chi' = \chi^{-1} \cdot |\cdot|^{-k_0/2} \). Note that \( \chi' \) has finite order. Extend \( \chi' \) to a Hecke character \( \chi' \) and twist \( f \) by \( \chi' \) to get an eigenform \( f \otimes \chi' \). Let \( f' \) denote the newform in the eigenspace spanned by \( f \otimes \chi' \) and let \( \pi' \) denote the corresponding automorphic representation. Then the \( v \)-component of \( \pi' \) is a subquotient of \( \pi(\chi \chi'_v \cdot |\cdot| \chi' \chi'_v) \). In particular, \( \chi \chi' \) is unramified at \( v \), so \( v \) divides the level \( n' \) of \( f' \), exactly once.

Suppose \( \lambda \) does not divide \( 2(q^2 - 1) \) and suppose that

\[
H^0(G_v, \bar{e} \otimes \text{ad}^0 \bar{\rho}_{f,\lambda}) \neq 0.
\]

Then Lemma 3.4 implies that \( \bar{\rho}_{f,\lambda}|_{G_v} \otimes \bar{k}_{f,\lambda} \cong \bar{\varepsilon} \bar{\lambda} \otimes \bar{\chi}_\lambda \). This means that

\[
\bar{\rho}_{f',\lambda}|_{G_v} \otimes \bar{k}_{f,\lambda} \cong (\bar{\rho}_{f,\lambda} \otimes \bar{\chi}_\lambda)|_{G_v} \otimes \bar{k}_{f,\lambda} \cong \bar{\varepsilon}^{1-k_0/2} \otimes \bar{\varepsilon}^{-k_0/2},
\]

so \( \bar{\rho}_{f',\lambda}|_{G_v} \otimes \bar{k}_{f,\lambda} \) is unramified at \( v \). Since \( \lambda \) does not divide \( q^2 - 1 \), we have that \( N_{F'/Q}(v) \neq 1 \mod \ell \), so we may apply [12, Theorem 0.1] to get a congruent eigenform \( f'' \) of level \( n'' \). That is, we get a set of Hecke eigenvalues \( \{a(m, f'')\} \) such that \( a(q, f'') \equiv a(q, f') \mod \lambda \) for all \( q \) not dividing \( n'' \ell \). By strong multiplicity one, there are only finitely many sets of eigenvalues, each one corresponding to a
newform of level dividing \( n'/v \). Therefore, if (3.6.1) holds for infinitely many \( \lambda \) then for some newform \( g \) of level dividing \( n'/v \) and for all \( q \) not dividing \( n' \),

\[
a(q, g) \equiv a(q, f') \mod \lambda
\]

for infinitely many \( \lambda \). We conclude that \( a(q, g) = a(q, f') \) for all \( q \) not dividing \( n' \), so applying strong multiplicity one again shows that \( g = f' \), a contradiction. \( \square \)

4. Local invariants for \( \ell = p \)

We now recall the theory of Fontaine-Laffaille. Let \( K \) be a finite unramified extension of \( \mathbb{Q}_\ell \) and let \( E/\mathbb{Q}_\ell \) be another finite extension containing \( K \). Let \( \sigma \) be the Frobenius automorphism on \( K \). Given an \( E \)-linear representation \( V \) of \( G_K \), define the finite free \( E \otimes_{\mathbb{Q}_\ell} K \)-module

\[
D_{\text{crys}}(V) = (B_{\text{crys}} \otimes_{\mathbb{Q}_\ell} V)^{G_K}
\]

where \( B_{\text{crys}} \) is Fontaine’s crystalline period ring. Note that \( D_{\text{crys}}(V) \) comes with a decreasing filtration \( \{D_{\text{crys}}(V)^i\}_i \) such that

\[
\cap_i D_{\text{crys}}(V)^i = 0 \text{ and } \cup_i D_{\text{crys}}(V) = D_{\text{crys}}(V).
\]

In addition, \( D_{\text{crys}}(V) \) comes with a \( 1_E \otimes \sigma \)-semilinear map \( \varphi : D_{\text{crys}}(V) \to D_{\text{crys}}(V) \). Call \( V \) crystalline if \( \dim_E V \) equals the rank of \( D_{\text{crys}}(V) \) as a \( E \otimes_{\mathbb{Q}_\ell} K \)-module.

Suppose \( V \) is an \( E \)-linear crystalline \( G_K \)-representation with Hodge-Tate filtration in the interval \( [(a+\ell-1), -a] \). Consider the category \( \text{MF}^{a,a+\ell}(\mathcal{O}_E) \) of strongly divisible lattices in \( D_{\text{crys}}(V) \) whose objects consist of finite free \( \mathcal{O} := \mathcal{O}_E \otimes \mathbb{Z} \), \( \mathcal{O}_K \)-lattices \( L \subset D_{\text{crys}}(V) \) with a filtration \( \{L^i := L \cap D_{\text{crys}}(V)^i\}_i \) and \( 1_{\mathcal{O}_E} \otimes \sigma \)-semilinear maps \( \{\varphi^L_i : L^i \to L\}_i \) such that

1. \( L^i \supset L^{i+1}, L^a = L, L^{a+\ell} = 0 \) and each \( L^i \) is a direct summand of \( L \),
2. \( \varphi^L_i|_{L^{i+1}} = \ell \varphi^L_{i+1} \) and \( L = \sum_i \varphi^L_i(L^i) \).

Then Fontaine-Laffaille [8] gives an equivalence of categories between \( \text{MF}^{a,a+\ell}(\mathcal{O}_K) \) and the category of \( \mathcal{O}_E[G_K] \)-modules that are finitely generated subquotients of \( E \)-linear crystalline \( G_K \)-representations \( V \) with Hodge-Tate weights in the interval \( [-(a+\ell-1), -a] \).

Remark 4.1. Here we use the definition of the Tate twist of a strongly divisible lattice as in Section 4 of [1] to extend the results of [8] to the case where \( a \neq 0 \).

Example 4.2. Let \( \psi : G_K \to \mathcal{O}_E \) be an unramified character of finite order and let \( \mathcal{O}_E(\psi) \) denote the \( \mathcal{O}_E[G_K] \)-module of rank one with \( G_K \)-action given by \( \psi \). Then the strongly divisible lattice \( D_\psi \) corresponding to \( \mathcal{O}_E(\psi) \) can be described as a free rank one \( \mathcal{O} \)-module such that \( L_0 = L_\psi, L_1^a = 0 \) and \( \varphi^L_0 \) is multiplication by some \( u \in \mathcal{O}^\times = (\mathcal{O}_E^K[\mathbb{Q}:\mathbb{Q}_\ell])^\times \). Denote this \( u \) by \( \psi(\sigma) \).

We adopt this notation since over some finite extension of \( E \), we have that \( L_\psi \) is isomorphic to a strongly divisible lattice \( L \) where \( \varphi^L_0 \) is multiplication by \( (\psi(\sigma), 1, \ldots, 1) \) (see [7]). In any case, the precise value of \( u \) will not be important for our intended application.

For the remainder of the section, we assume \( \ell \) is unramified in \( F \) (a totally real extension of \( \mathbb{Q} \) of degree \( d \)) and set \( K = F_\nu \) for a place \( \nu \) of \( F \) dividing \( \ell \).
Example 4.3. Let $f$ be a newform on $F$ in $S_k(n, \psi)$. Set $k_0 = \max_i \{k_i\}$. For a prime $v|\ell$ of $F$, let $E = K_{f,\lambda}F_v$ and consider the Galois representation $\rho_{f,\lambda}|G_v : G_v \to \text{GL}_2(E)$. Let $V_{f,\lambda}$ be a 2-dimensional $E$ vector space on which $G_v$ acts by $\rho_{f,\lambda}|G_v$. The fixed embedding $\iota_v : Q \hookrightarrow Q_\ell$ gives a partition of the different embeddings $\tau_i$ as a disjoint union of subsets $J_v$ where denotes the subset of $\tau_i$ that give the different embeddings of $F_v$ into $Q_\ell$ after composition with $\iota_v$. Note that $V_{f,\lambda}$ is crystalline with labeled Hodge-Tate weights $\lambda$. This is not enough to completely identify $V_{f,\lambda}$.

Then there is an $O_{E}$-basis $x, y$ of $V_{f,\lambda}$ with a 1 in its $i$th component and zeroes everywhere else. Finally, define
\[
d_{\geq i} = \sum_{j \in \cup_{k \geq i} T_k} e_j \quad \text{and} \quad d_{<i} = \sum_{j \in \cup_{k < i} T_k} e_j.
\]

Then there is an $O$ basis $x, y$ of $L_f$ such that the filtration satisfies:
\[
L_{f,\lambda}^i = \begin{cases} 
O_x \oplus \mathcal{O}_y, & \text{for } i \leq \frac{k_0-k_0}{2}, \\
O_x \oplus \mathcal{O}_{d_{\geq i}y}, & \text{for } \frac{k_0-k_0}{2} + 1 \leq i \leq \frac{k_0-k_0}{2}, \\
O_x \oplus \mathcal{O}_{d_{<i}y}, & \text{for } \frac{k_0-k_0}{2} + 1 \leq i \leq \frac{k_0-k_0}{2}, \\
\vdots & \text{for } \frac{k_0-k_0}{2} + 1 \leq i \leq \frac{k_0-k_0}{2}, \\
O_{d_{<i}x}, & \text{for } \frac{k_0-k_0}{2} + 1 \leq i \leq \frac{k_0-k_0}{2}, \\
\vdots & \text{for } \frac{k_0-k_0}{2} + 1 \leq i \leq \frac{k_0-k_0}{2}, \\
O_{d_{<i}x}, & \text{for } \frac{k_0-k_0}{2} + 1 \leq i \leq \frac{k_0-k_0}{2}, \\
0, & \text{for } i \geq \frac{k_0-k_0}{2} + 1.
\end{cases}
\]

This is not enough to completely identify $L_{f,\lambda}$ up to isomorphism, but it will be enough for our purposes.

We fix some notation for use in Proposition 4.4. Let
\[
\varphi_{f,\lambda}^L(x) = \alpha x + \beta y
\]
for some $\alpha, \beta$ in $O$. So writing $\alpha = (\alpha_\tau)_{\tau \in J_v}$ and $\beta = (\beta_\tau)_{\tau \in J_v}$, we conclude that the $\lambda$-adic valuations $v_\lambda(\alpha_\tau)$ and $v_\lambda(\beta_\tau)$ are at least $\frac{k_0-k_0}{2} + k_0$ for all $\tau$ by condition 2 of the definition of the objects of $\text{MF}^{0,\ell}(O_{F_v})$. Here we normalized $v_\lambda$ so that $v_\lambda(1) = 1$.

Proposition 4.4. Suppose $f$ is a Hilbert newform on $F$ of weight $k = (k_1, \ldots, k_\alpha)$, level $n$, and character $\psi$. Assume that $k_0 > 2$. Then for $\ell \geq 2k_0 + 1$, unramified in $F$ and prime to $n$,
\[
H^0(G_v, \tilde{\varepsilon} \otimes \text{ad}^0 \tilde{\rho}_{f,\lambda}) = 0.
\]

Proof. Since $\det \rho_{f,\lambda} = \psi_f^{-1} \varepsilon^{1-k_0}$, we have the Galois-stable lattice
\[
\text{ad}^0 T_{f,\lambda}(1) \cong (\text{Sym}^2(T_{f,\lambda}) \otimes \mathcal{O}_E(\psi_f))(h_0).
\]
we have

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or equivalently,

But we also know that

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CM and that it is not a twist of a base change of a Hilbert newform on

Let

Theorem 1.1.

As we assumed

unobstructed for almost all

Then

We are now ready to prove Theorem 1.1.

\[ L^0 = (\text{Sym}^2(L_{f,\lambda}) \otimes \mathcal{O} L_{\psi_f})(k_0). \]

Note that by [1, Lemma 4.5],

\[
(4.4.1) \quad H^1(G_{\psi,\bar{\mathcal{E}}} \otimes \text{ad}^0 \rho_{f,\lambda}) \cong \ker(1 - \psi_f^L : L^0/\lambda L^0 \to L/\lambda L).
\]

Furthermore, by the definition of Tate twists for strongly divisible lattices,

\[
L^0 = (\text{Sym}^2(L_{f,\lambda}) \otimes \mathcal{O} L_{\psi_f})^{k_0} = \left\{ v \otimes w \mid v = \sum_i a_i(u \otimes u') \in \text{Sym}^2(L_{f,\lambda}), u \in L_{f,\lambda}, u' \in L_{f,\lambda}, i + i' = k_0 \right\}
\]

\[
= L_{f,\lambda}^{k_0-2+k_0^v} \otimes \mathcal{O} L_{f,\lambda}^{k_0-2+k_0^v} \otimes \mathcal{O} L_{\psi_f}^0
\]

where \( w \) is a generator of the rank one \( \mathcal{O} \)-module \( L_{\psi_f} \). Set \( v = x \otimes x \otimes w \). Then we have

\[
\psi_f^L(ad_{<1} v) = a^\sigma d_{<1}^\sigma \psi_f^L(x) \otimes \varphi_{k_0-2+k_0^v}^L(x) \otimes \varphi_0^L(w)
\]

(4.4.2)

where the superscript \( \sigma \) denotes the action of \( 1_{\mathcal{O}_E} \otimes \sigma \) on the given element of \( \mathcal{O} \). Suppose that (4.4.1) is nonzero. Thus if \( ad_{<1}(x \otimes x \otimes w) \) is a nonzero element of the kernel of \( 1 - \psi_f^L \) then (4.4.2) implies that there is some \( \tau \in J, \sigma \) such that

\[
\frac{(ad_{<1})^\sigma \psi_f^L(x) \alpha^\tau}{\ell^{k_0-2+k_0^v-1}} \equiv (ad_{<1}), \not\equiv 0 \mod \lambda.
\]

This implies that the \( \lambda \)-adic valuation of the numerator is \( \frac{k_0-2+k_0^v}{2} + 1 \). Since \( v_\lambda(a_\tau \psi_f^L(\sigma)) = 0 \), this means that

\[
v_\lambda(\alpha^\tau) = \frac{k_0-2+k_0^v}{2} + 1.
\]

But we also know that \( v_\lambda(\alpha_\tau) \geq \frac{k_0-2+k_0^v}{2} \), so we have that \( k_0-2+k_0^v \leq \frac{k_0-2+k_0^v+1}{2} \), or equivalently,

\[
k_0 - 2 + k_0^v \leq 2.
\]

As we assumed \( k_0 > 2 \), this proves the proposition. \( \square \)

We are now ready to prove Theorem 1.1.

**Theorem 1.1.** Let \( f \) be as in Proposition 4.4 and suppose that it does not have CM and that it is not a twist of a base change of a Hilbert newform on \( E \subsetneq F \). Then \( H^2(G_{F,S}, \text{ad}^0 \rho_{f,\lambda}) = 0 \) for almost all primes \( \lambda \) of \( K_f \). That is, \( D_{\rho_{f,\lambda}}^{\text{det}=\delta} \) is unobstructed for almost all \( \lambda \).
Proof. We verify that the hypotheses (1) – (3) of Proposition 2.4 hold for almost all \( \lambda \). Combining the results of Corollary 3.2, Proposition 3.6 and Proposition 4.4 shows that \( H^0(G_v, \xi \otimes \text{ad}^0 \bar{\rho}_{f, \lambda}) = 0 \) for almost all \( \lambda \). The Selmer group \( H^1_f(G_F, A_{\rho_{f, \lambda}}) \) vanishes for almost all \( \lambda \) because of \([5, \text{Theorem 6.6}], [6, \text{Theorem 2.1}]\) and the fact that (in Dimitrov’s notation)
\[
H^1_f(G_F, A_{\rho_{f, \lambda}}) \subset H^1_\Sigma(G_F, A_{\rho_{f, \lambda}})
\]
for any finite set of primes \( \Sigma \).

Theorem B(i) of \([5]\) and [6, Theorem 2.1] tell us that \( H^1_f(G_F, V_{\rho_{f, \lambda}}) = 0 \) for almost all \( \lambda \). To show the vanishing of \( H^1_f(G_F, V_{\rho_{f, \lambda}}(1)) \), we define for a place \( v|\ell \) of \( F \) the tangent space
\[
t_V = ((B_{\text{crys}}/B_{\text{crys}}^+) \otimes \mathbb{Q}_\ell V)^{G_{F_v}}
\]
of a crystalline \( E \)-linear representation \( V \) where \( E \) is a finite extension of \( \mathbb{Q}_\ell \) containing \( K_{f, \lambda} \) and \( F_v \). Then \([10, \text{Proposition I.2.2.2(ii)}] \) tells us that
\[
t_V \cong D_{\text{crys}}(V)/D_{\text{crys}}(V)^0.
\]
In particular, for \( V_{\rho_{f, \lambda}} \), we extend scalars to a finite extension \( E \) of \( \mathbb{Q}_\ell \) such that \( E \) contains \( F_v \) for all places \( v|\ell \) in \( F \) and set
\[
t_{V_{\rho_{f, \lambda}}} = \bigoplus_{v|\ell} t_{V_{\rho_{f, \lambda}}|G_{F_v}}.
\]
By Schur’s lemma, \( H^0(G_F, \xi \otimes \text{ad}^0 \bar{\rho}_{f, \lambda}) = 0 \), so \([10, \text{Remark II.2.2.2}] \) implies that
\[
\dim_E H^1_f(G_F, V_{\rho_{f, \lambda}}(1)) = \dim_E H^1_f(G_F, V_{\rho_{f, \lambda}}) = -\dim_E t_{V_{\rho_{f, \lambda}}} + \sum_{v|\infty} \dim_E H^0(G_v, V_{\rho_{f, \lambda})).
\]
A straightforward computation using the Hodge-filtration on \( V_{\rho_{f, \lambda}} \) shows that the right-hand-side vanishes, so \( \dim_E H^1_f(G_F, V_{\rho_{f, \lambda}}(1)) = \dim_E H^1_f(G_F, V_{\rho_{f, \lambda}}) \). Thus Proposition 2.4 implies the unobstructedness of \( D^{\text{det}=\delta}_{\rho_{f, \lambda}} \) for almost all \( \lambda \). □

5. Explicit Computations

The methods we used to prove Theorem 1.1 are essentially effective in the sense that given enough information about the Hecke eigenvalues of a given Hilbert newform as well as the eigenvalues of the other newforms of the same level, one can find an explicit lower bound \( B \) such that for all \( \ell \geq B \), the deformation problem \( D^{\text{det}=\delta}_{\rho_{f, \lambda}} \) is unobstructed for all \( \lambda \) over these \( \ell \). We illustrate this with an example.

Let \( F = \mathbb{Q}(\sqrt{5}) \), \( k = (4, 8) \), and \( n = (1) \). Then using MAGMA we computed that the space of cuspforms \( S_k(n) \) is one dimensional. Thus \( S_k(n) \) is generated by a newform \( f \) whose first few Hecke eigenvalues \( c(f, \mathfrak{n}) \) we computed in MAGMA and list in Table 1.

<table>
<thead>
<tr>
<th>Table 1. Hecke eigenvalues of ( f )</th>
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<tbody>
<tr>
<td>( N_F/\mathbb{Q}(\pi) )</td>
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<td>( c(f, \mathfrak{n}) )</td>
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<tr>
<td>( N_F/\mathbb{Q}(\pi) )</td>
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<td>( c(f, \mathfrak{n}) )</td>
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</table>
Remark 5.1. It can be shown that $K_f = \mathbb{Q}(\sqrt{5})$ in this case.

Remark 5.2. In what follows, note that $S = \{\lambda\} \cup \{v|\infty\}$ for $\bar{\rho}_{f,\lambda}$.

Proposition 5.3. The deformation problem $D_{\bar{\rho}_{f,\lambda}}^{\text{det} = \delta}$ is unobstructed for all primes $\lambda$ of $K_f$ over $\ell \geq 17$.

Proof. Our approach is to give a lower bound on $\ell$ for which the residual representation $\bar{\rho}_{f,\lambda}$ is absolutely irreducible and for which the three hypotheses of Proposition 2.4 hold. We begin with absolute irreducibility. Let $\omega = \frac{3 + \sqrt{5}}{2}$. By [4, Proposition 3.1(ii)], since $\omega$ is a totally positive unit, we conclude that $\bar{\rho}_{f,\lambda}$ is absolutely irreducible for all $\lambda$ not dividing $N_F/\mathbb{Q}(\omega - 1) = -1$, $N_F/\mathbb{Q}(\omega^2 - 1) = -5$, $N_F/\mathbb{Q}(\omega^8 - 1) = -2205$, and $N_F/\mathbb{Q}(\omega^9 - 1) = -5776$. More concretely, computing the prime factors of the principal ideals generated by these elements, [4, Proposition 3.1(ii)] tells us that $\bar{\rho}_{f,\lambda}$ is absolutely irreducible for all $\lambda$ over $\ell \geq 17$ except possibly the primes over 19.

Note that since $\bar{\rho}_{f,\lambda}$ is an odd representation, it is absolutely irreducible if and only if it is irreducible. Thus to prove absolute irreducibility of $\bar{\rho}_{f,\lambda}$ for the remaining $\lambda$, it suffices to provide a prime $p$ over $p \neq 19$ such that the characteristic polynomial of $\bar{\rho}_{f,\lambda}(\text{Frob}_p)$ is irreducible over $k_{f,\lambda}$. Recall that the characteristic polynomial for $\bar{\rho}_{f,\lambda}(\text{Frob}_p) = X^2 - c(f,p)X + p^3$ if $p$ splits in $F$. In particular, for each $\lambda$ over 19, we found a prime $p$ over 11 such that the polynomial $X^2 - c(f,p)X + 11^3$ is irreducible over $k_{f,\lambda}$. Thus $\bar{\rho}_{f,\lambda}$ is absolutely irreducible for all $\lambda$ over $\ell \geq 17$.

We now check hypotheses (1)–(3) of Proposition 2.4. For (1), we know that $H^0(G_{\lambda}, \mathbb{Z} \otimes \text{ad}^0 \bar{\rho}_{f,\lambda}) = 0$ for all $\lambda$ over $\ell \geq 15$ by Proposition 4.4. Regarding (2), as we discussed in the proof of Theorem 1.1, the vanishing of $H^1_f(G_F, V_{\bar{\rho}_{f,\lambda}}(1))$ is equivalent to the condition that $H^1_f(G_F, V_{\bar{\rho}_{f,\lambda}}) = 0$. Furthermore, Dimitrov [5, Theorem B] showed that $H^1_f(G_F, V_{\bar{\rho}_{f,\lambda}})$ vanishes as long as $\ell \geq 13$, and the image of $\text{Ind}_F^G \bar{\rho}_{f,\lambda}$ is “large.” (We will give more details about this large image condition in the next paragraph.) This means that for the desired set of primes $\lambda$, whenever this “large image” condition holds, hypothesis (2) of Proposition 2.4 also holds. Moreover, let $\eta_f$ denote the congruence ideal obtained from the $\mathfrak{o}$-algebra homomorphism $T_m \rightarrow \mathfrak{o}$ by $T_a \mapsto \iota_a(c(f,a))$ where $\mathfrak{o} = \mathcal{O}_{K_{f,\lambda}}$, $T$ is the Hecke algebra $\mathfrak{o}[T_a | a \in \mathcal{O}_F]$ and $\iota_a$ is the fixed isomorphism $\mathbb{C} \rightarrow K_{f,\lambda}$ extending the embedding $\mathcal{O}_{K_{f,\lambda}} \rightarrow \mathfrak{o}$. (See [5, Definition 3.1] as well as the discussions before [5, Theorems 1.4 and 3.6] for more details about $\eta_f$.) Then [5, Theorem 3.6] implies that for $\ell \geq 13$, and $\text{Ind}_F^G \bar{\rho}_{f,\lambda}$ satisfying the same large image hypothesis, the Selmer group $H^1_f(G_F, A_{\bar{\rho}_{f,\lambda}}) = 0$ if and only if $\eta_f = \mathfrak{o}$. That is, if and only if $\lambda$ does not divide $\eta_f$. By definition, however, $\lambda$ divides $\eta_f$ if and only if there is another newform $g$ of the same weight, level and character such that $c(f,a) \equiv c(g,a) \mod \lambda$ for all $a \in \mathcal{O}_F$. As $f$ is the only newform in $S_{\lambda}(n)$, this means that $H^1_f(G_F, A_{\bar{\rho}_{f,\lambda}}) = 0$ for all such $\lambda$. That is, hypothesis (3) also holds for $\lambda$ over $\ell \geq 13$, satisfying the large image condition. Thus we are reduced to checking that this large image condition on $\text{Ind}_F^G \bar{\rho}_{f,\lambda}$ holds for all $\lambda$ over $\ell \geq 17$.

The large image condition on $\text{Ind}_F^G \bar{\rho}_{f,\lambda}$ that we referred to throughout the previous paragraph is a somewhat technical hypothesis that Dimitrov uses for Theorem 1.4 of [5]. We refer the interested reader to [5, Theorem A] for a detailed statement.
of this large image hypothesis on $\text{Im}^Q \bar{\rho}_{f,\lambda}$. In our case, however, since the weight (4,8) is non-induced in the sense of [4, Definition 3.11] and we assume that $\ell \geq 17$, we may instead use the large image condition on $\bar{\rho}_{f,\lambda}$ that $\text{Im}(\bar{\rho}_{f,\lambda})$ contains a conjugate of $\text{SL}_2(k_{f,\lambda})$ (see [4, Proposition 3.13]). Moreover, since we have already shown that $\bar{\rho}_{f,\lambda}$ is irreducible for all $\lambda$ over $\ell \geq 17$, we can use Dickson’s classification of subgroups of $\text{GL}_2(k_{f,\lambda})$ in such cases. In particular, this classification states that an irreducible subgroup of $\text{GL}_2(k_{f,\lambda})$ that does not contain a conjugate of $\text{SL}_2(k_{f,\lambda})$ is isomorphic to either a dihedral group or one of $A_4, S_4,$ or $A_5$.

Thus we need to show that the projective image of $\text{Im}(\bar{\rho}_{f,\lambda}) \subset \text{GL}_2(k_{f,\lambda})$ is not isomorphic to a dihedral group nor any of the groups $A_4, S_4,$ and $A_5$.

To check that the projective image of $\text{Im}(\bar{\rho}_{f,\lambda})$ is not dihedral, we use [4, Lemma 3.4]. More specifically, assume that the image of $\bar{\rho}_{f,\lambda}$ in $\text{PGL}_2(k_{f,\lambda})$ is dihedral, meaning $\bar{\rho}_{f,\lambda} \cong \bar{\rho}_{f,\lambda} \otimes \chi_{K/F}$ where $\chi_{K/F}$ is the character of a quadratic extension $K/F$. Then suppose that $\ell \neq 2k_{\tau_i} - 1$ for all $i$ where $k = (k_{\tau_1}, \ldots, k_{\tau_d})$ is the weight of $f$ and $d = [F : \mathbb{Q}]$, this lemma says that $K/F$ is unramified outside of $\mathfrak{n}$. In our case, $\mathfrak{n} = \mathcal{O}_F$. Since $F$ has class number one, we conclude that no such $K$ exists. Hence the image of $\bar{\rho}_{f,\lambda}$ in $\text{PGL}_2(k_{f,\lambda})$ is not dihedral.

Finally, to show that the projective image of $\bar{\rho}_{f,\lambda}$ is not isomorphic to $A_4, S_4,$ or $A_5$, we use Section 3.2 of [4]. The main result of this section is that if

$$\ell - 1 > \frac{5}{d} \sum_{i=1}^{d} k_{\tau_i} - 1,$$

then the projective image of $\bar{\rho}_{f,\lambda}$ is not isomorphic to any of the groups $A_4, S_4,$ or $A_5$. In our case, $d = 2$ and $k = (4,8)$ so it is easy to conclude that the image of $\bar{\rho}_{f,\lambda}$ is not isomorphic to $A_4, S_4,$ or $A_5$ for $\lambda$ over $\ell \geq 25$. For $\lambda$ over 19, a closer analysis of [4, Section 3.2] shows that if the projective image of $\bar{\rho}_{f,\lambda}$ is $A_4, S_4,$ or $A_5$ then the arguments there imply that either $\pm 3$ or $\pm 7$ has order $\leq 5$ in $\mathbb{Z}/18\mathbb{Z}$, which is a contradiction. Similarly for $\lambda$ over 17, these same arguments imply that 58, 116, 122, or 244 would have order at most five in $\mathbb{Z}/288\mathbb{Z}$ if the projective image of $\bar{\rho}_{f,\lambda}$ is $A_4, S_4,$ or $A_5$, which is false. For $\lambda$ over 23, if the projective image of $\bar{\rho}_{f,\lambda}$ is $A_4, S_4,$ or $A_5$ then 76, 164, 370, or 466 would have order at most five in $\mathbb{Z}/528\mathbb{Z}$, which is also a contradiction.

\section*{References}


