Math 333 - Differential Equations

Optional: How slope of the line at equilibrium solution affects sketch of phase line or solution curves

Short answer: Qualitatively there's no difference. Analytically, there is a difference.

We'll consider two examples: \( \frac{dy}{dt} = f(y) = y \) and \( \frac{dy}{dt} = g(y) = y^3 \). Both of these differential equations have an source equilibrium at \( y = 0 \), however \( g(y) = y^3 \) has the condition that \( g'(0) = 0 \), and thus the equilibrium can not be classified using the Linearization Test.

Graphs of \( f(y) \) and \( g(y) \):

\[
\begin{align*}
\frac{df}{dy} & = y \\
\frac{dg}{dy} & = y^3
\end{align*}
\]

Phaselines:

\[
\begin{align*}
\frac{dy}{dt} & = y \\
\frac{du}{dt} & = y^3
\end{align*}
\]

Now let's look at solutions. \( \frac{dy}{dt} = y \) is an exponential growth DE, and the general solution is \( y(t) = Ce^t \). We can solve \( \frac{dy}{dt} = y^3 \) by separation of variables.

\[
\begin{align*}
\frac{dy}{y^3} & = dt \\
\int dy & = \int dt \\
-\frac{1}{2}y^{-2} & = t + c \\
y^2 & = -2t + c \\
y & = \pm \frac{1}{\sqrt{c - 2t}}
\end{align*}
\]
Note that the solutions to this DE are defined only for \( t < \frac{e}{2} \) while solutions for the exponential growth DE are defined for all \( t \).

In both cases, solutions are qualitatively the same, as you can see by the fact that the phase lines are the same. The solutions increase to \( \infty \) for \( y > 0 \), decrease to \( -\infty \) for \( y < 0 \) and there's an equilibrium solution at \( y = 0 \). However in the case of \( \frac{dy}{dt} = y^3 \) solutions blow up to \( \pm \infty \) at a finite time.

Here are the solution curves for both differential equations, through the same set of initial conditions.

Notice that the solutions to \( \frac{dy}{dt} = y^3 \) blow up faster (in finite time) and take longer to go to 0 as \( t \to -\infty \).