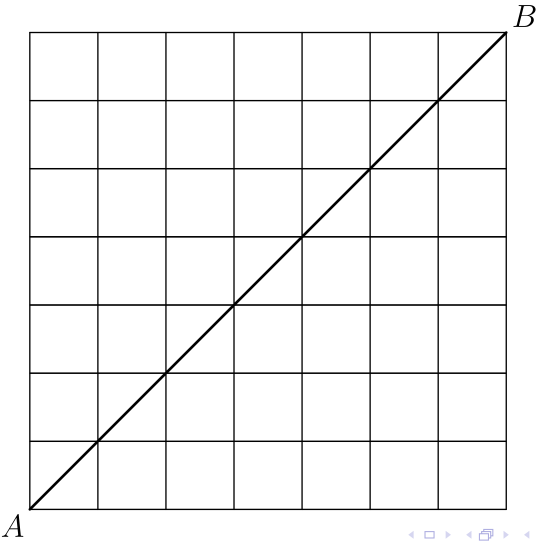
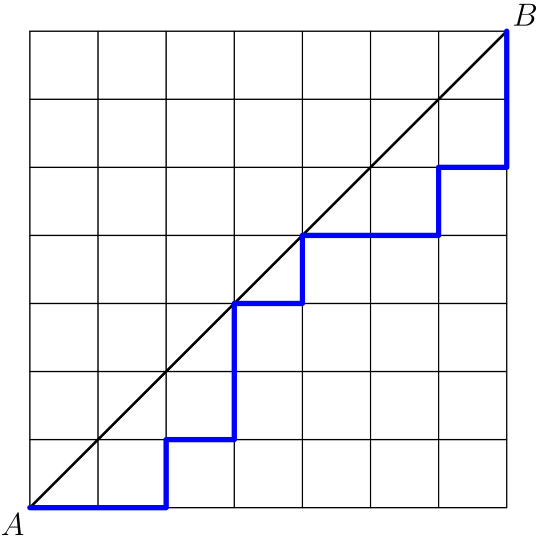
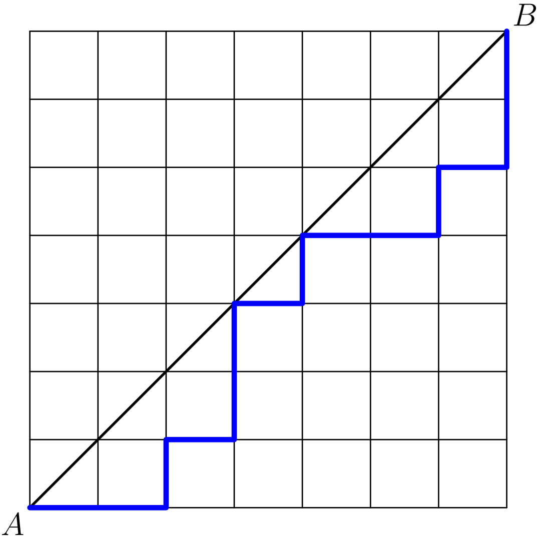
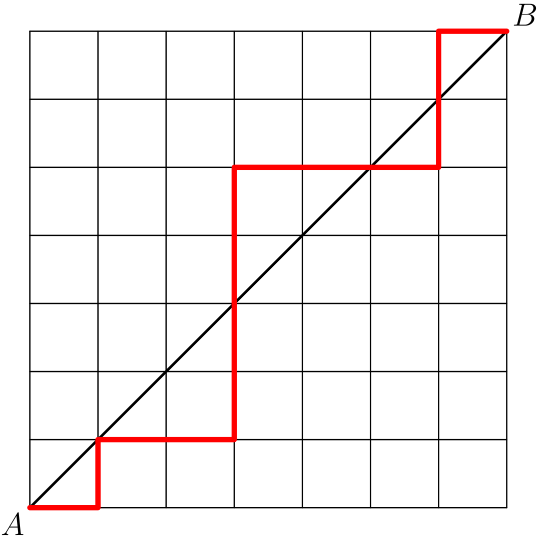
How many shortest paths from A to B do *not* pass above the diagonal?

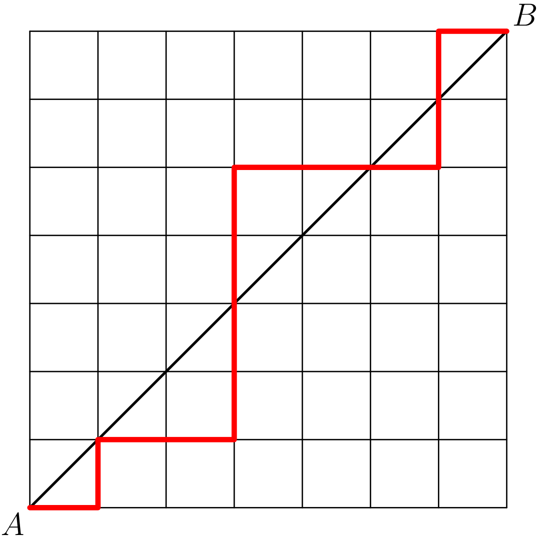



B
A



Blue path is good


B
A



Red path is bad

Number of good paths

$$\text{Number of good paths} = \text{Total Number of paths} - \text{Number of bad paths}$$

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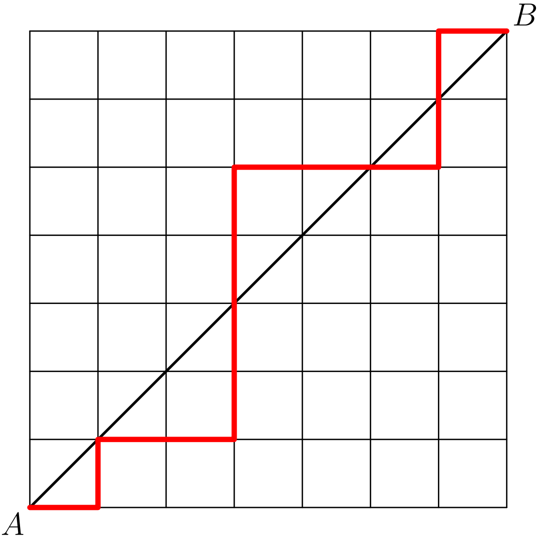
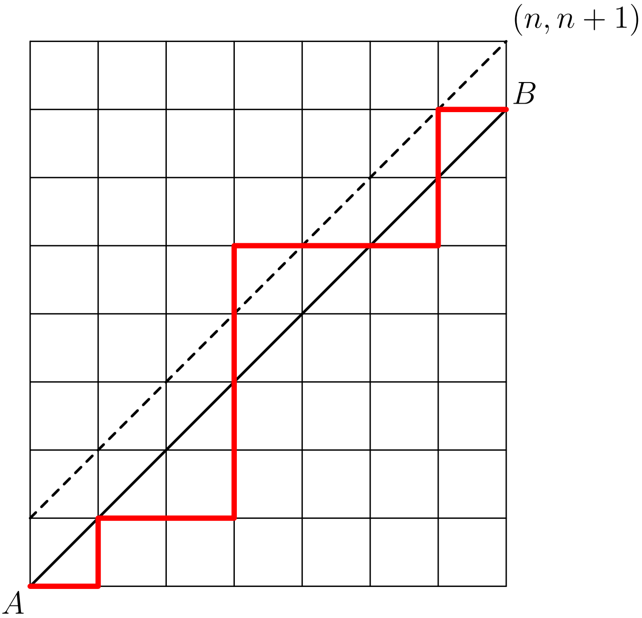
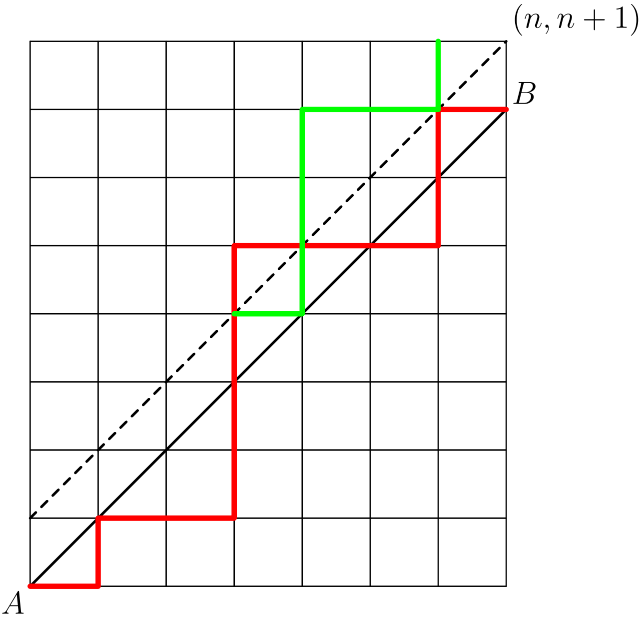
So it is sufficient to count the number of bad paths.

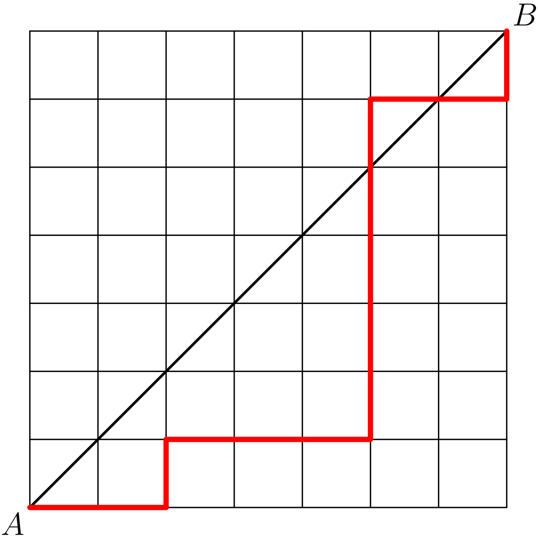
How to count the number of bad paths?

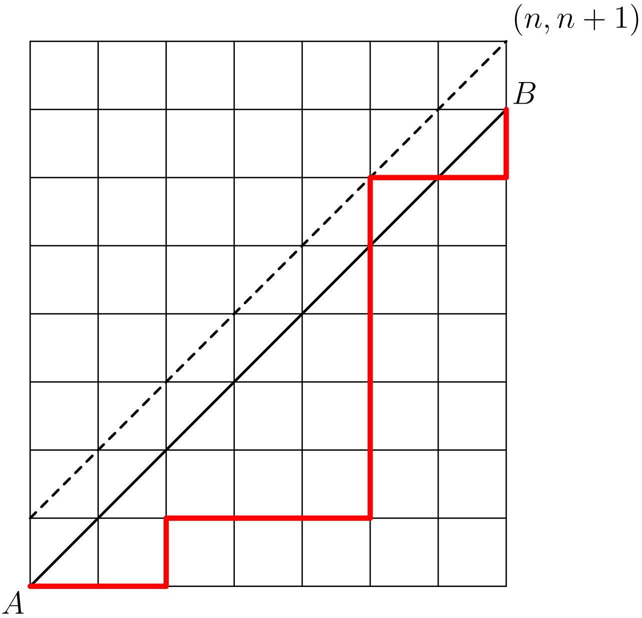


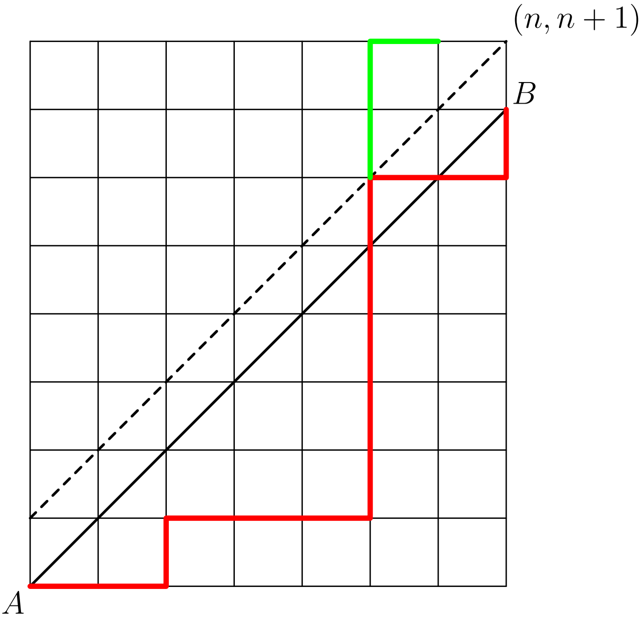

$(n, n+1)$
B
A



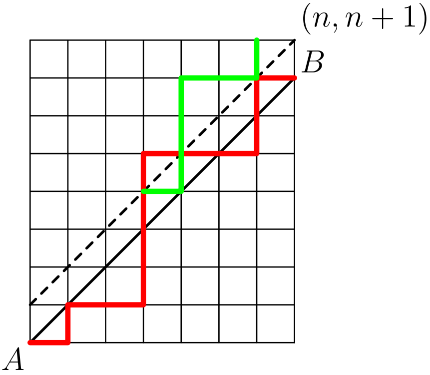
$(n, n+1)$
B
A

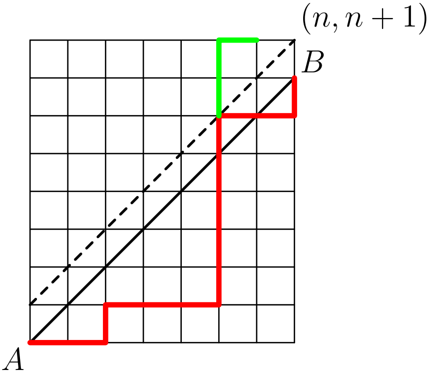

B
A


(n, n + 1)
B
A



$(n, n+1)$
B
A





In fact, reflection turns every bad path into a path reaching $(n-1, n+1)$.

Moreover, every path reaching $(n-1, n+1)$ is obtained from a bad path.

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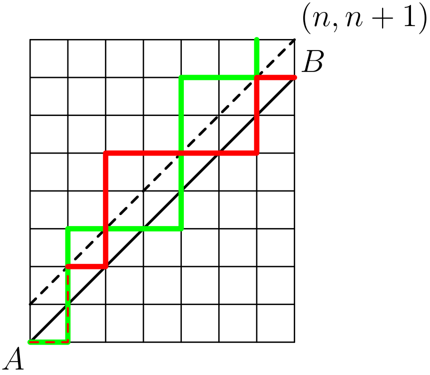

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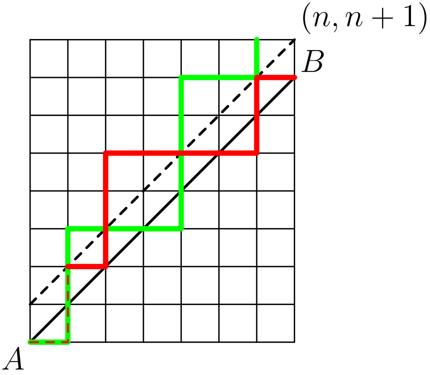

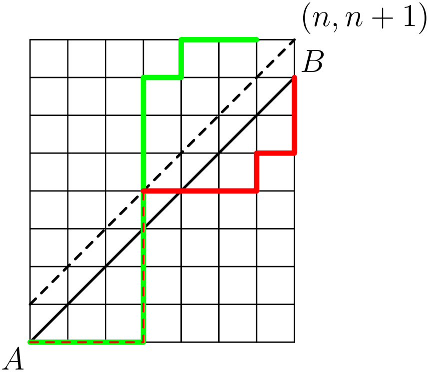

Conclusion: There is a one-to-one correspondence between the set of bad paths and the set of paths reaching $(n-1, n+1)$.

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Finally,

Number of good paths

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Finally,

Number of good paths $=$ Total Number of paths – Number of bad paths

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$$\begin{aligned} \text{Number of good paths} &= \text{Total Number of paths} - \text{Number of bad paths} \\ &= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n-1)!(n+1)!} \end{aligned}$$

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&= \frac{1}{n+1}\binom{2n}{n}
\end{aligned}$$

The number $C_n = \frac{1}{n+1}\binom{2n}{n}$ is called the n^{th} Catalan number and has a lot of applications.