Central Sets in Commutative Adequate Partial Semigroups

Jillian McLeod[1]

Abstract. Adequate partial semigroups were first introduced by Bergelson, Blass, and Hindman. They give rise to a semigroup δS, which is a subset of βS. We extend the Commutative Central Sets Theorem to adequate partial semigroups. We also show that if S is an adequate countable cancellative semigroup, δS contains many copies of $\mathbb{H}$ in δS.

1. Introduction

The notion of *central* subsets of $\mathbb{N}$ was introduced by Furstenberg [3] in terms of dynamical systems. The definition given there of "central" makes sense in any semigroup, S, and was shown [7] to be equivalent to a simpler algebraic characterization which we use below. This algebraic characterization is in the setting of the Stone–Čech compactification, βS, of the semigroup S. It is this characterization that we use as our definition of central.

1.1 Definition. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. Then A is *central* if and only if there is some minimal idempotent $p \in \beta S$ with $p \in \overline{A}$.

In [3] Furstenberg also proves the powerful "Central Sets Theorem". The non-commutative version of the theorem is particularly complicated to state and for this reason we will discuss here only the commutative version. See [4, Section 14.4] for a statement and proof of the non-commutative version.

This paper is organized in the following way: In Section 1 we state the Commutative Central Sets Theorem for a semigroup. We also prove a more general statement of van der Waerden's theorem as a corollary. Section 2 provides basic background about adequate partial semigroups including the definition of δS. The fact that δS is a compact right topological semigroup in βS provides the context for our investigation of its central subsets. Please refer to [5] for more on the algebra of δS. In Section 3 we present our main result Theorem 3.3 which is the extension of the central sets theorem to commutative adequate partial semigroups. We end the section and the article by giving

---

[1] The author acknowledges support received from the Association for Women in Mathematics during the academic year 2003-2004. She also thanks the University of Hull for its hospitality while this research was being conducted.

some examples of adequate partial semigroups which produce copies of the semigroup $\mathbb{H} = \bigcap_{n\in\mathbb{H}} c\ell_{\beta\mathbb{N}}(2^n\mathbb{N})$.

1.2 Definition.

a) Φ is the set of all functions $f : \mathbb{N} \to \mathbb{N}$ for which $f(n) \leq n$ for all $n \in \mathbb{N}$.

b) Given a set A, $\mathcal{P}_f(A) = \{F : \emptyset \neq F\ F \subseteq A$ and F is finite$\}$.

c) Let $(S, \cdot)$ be a semigroup and let $\langle x_n\rangle_{n=1}^{\infty}$ be an infinite sequence in S. Then

$$FS(\langle x_n\rangle_{n=1}^{\infty}) = \{\sum_{n\in F} x_n :\ F \in \mathcal{P}_f(\mathbb{N})\}$$

$$\text{and } FP(\langle x_n\rangle_{n=1}^{\infty}) = \{\prod_{n\in F} x_n :\ F \in \mathcal{P}_f(\mathbb{N})\}.$$

1.3 Theorem (Central Sets Theorem). *Let S be a commutative semigroup, let A be a central subset of S, and for each $l \in \mathbb{N}$, let $\langle y_{l,n}\rangle_{n=1}^{\infty}$ be a sequence in S. There exist a sequence $\langle a_n\rangle_{n=1}^{\infty}$ in S and a sequence $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that* $\max H_n < \min H_{n+1}$ *for each $n \in \mathbb{N}$ and such that for each $f \in \Phi$, $FP(\langle a_n \cdot \prod_{t\in H_n} y_{f(n),t}\rangle_{n=1}^{\infty}) \subseteq A$.*

Proof. [4, Theorem 14.11].

An easily derivable consequence of the central sets theorem is the following extension of van der Waerden's Theorem, which says that given any sequence $\langle x_n\rangle_{n=1}^{\infty}$ and any central set A, there exist arbitrarily long arithmetic progressions in A whose increment comes from $FS(\langle x_n\rangle_{n=1}^{\infty})$, [3].

1.4 Corollary. *Let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$, let $r \in \mathbb{N}$, and let $\mathbb{N} = \bigcup_{i=1}^{r} A_i$. Then there is some $i \in \{1,\ 2,\ \ldots,\ r\}$ such that for all $l \in \mathbb{N}$ there exists $a \in \mathbb{N}$ and $d \in FS(\langle x_n\rangle_{n=1}^{\infty})$ with $\{a,\ a+d,\ a+2d,\ \ldots,\ a+ld\} \subseteq A_i$.*

Proof. A stronger statement is proved in [4, Exercise 14.3.1]. Pick $i \in \{1,\ 2, \ldots,\ r\}$ such that A_i is central in $(\omega,\ +)$ (where $\omega = \mathbb{N} \cup \{0\}$). For each $k,\ n \in \mathbb{N}$, let $y_{k,\ n} = (k-1)\cdot x_n$. Pick sequences $\langle a_n\rangle_{n=1}^{\infty}$ and $\langle H_n\rangle_{n=1}^{\infty}$ as guaranteed by the central sets theorem. Given $l \in \mathbb{N}$, pick any $m > l$ and let $a = a_m$ and let $d = \sum_{t\in H_m} x_t$. Now given $k \in \{0,\ 1, \ldots,\ l\}$, pick any $f \in \Phi$ such that $f(m) = k+1$. Then $a + kd = a_m + \sum_{t\in H_m} y_{f(m),\ t}$ which is in $FP(\langle a_n \cdot \sum_{t\in H_n} y_{f(n),t}\rangle_{n=1}^{\infty}) \subseteq A_i$. □

The author would like to thank Dona Strauss for inviting her to the University of Hull and for all her support, guidance and suggestions on this paper.

2. The definition of δS.

Given a set S, and a natural binary operation, it is often convenient to define the operation for only a subset of $S \times S$.

2.1 Definition. A *partial semigroup* is a pair $(S, *)$ where $*$ maps a subset of $S \times S$ to S and for all $a, b, c, \in S$, $(a * b) * c = a * (b * c)$ in the sense that if either side is defined, then so is the other and they are equal.

Some easy examples are:

1. Let $\mathcal{R} = \{A :$ there exist $m, n \in \mathbb{N}$ such that A is an $m \times n$ matrix with entries from $\mathbb{Z}\}$, with the usual matrix multiplication. We know that for an $m \times n$ matrix M and an $m' \times n'$ matrix N in $\mathcal{R}$, $M \cdot N$ is defined if and only if $n = m'$. So if we define $*$ as follows:
$$M * N = \begin{cases} M \cdot N & \text{if } n = m' \\ \text{undefined} & \text{otherwise} \end{cases}$$
then $(\mathcal{R}, *)$ is a partial semigroup.

2. Given a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in some semigroup $(S, \cdot)$, let $T = FP(\langle x_n \rangle_{n=1}^{\infty})$ where
$$FP(\langle x_n \rangle_{n=1}^{\infty}) = \{\textstyle\prod_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$$
and the products are taken in increasing order of indices. Then T is not likely to be closed under $\cdot$. On the other hand, if we let if we let $(\prod_{n \in F} x_n) * (\prod_{n \in G} x_n)$ be
$$\begin{cases} \prod_{n \in F \cup G} x_n & \text{if } \max F < \min G \\ \text{undefined} & \text{if } \max F \geq \min G \end{cases}$$
Then $(T, *)$ is a partial semigroup.

While both $(\mathcal{R}, *)$ and $(T, *)$ are partial semigroups, for our purposes, partial semigroups like $(T, *)$ are particularly interesting.

2.2 Definition. Let $(S, *)$ be a partial semigroup.

(a) For $s \in S$, $\varphi(s) = \{t \in S : s * t$ is defined$\}$.

(b) For $H \in \mathcal{P}_f(S)$, $\sigma(H) = \bigcap_{s \in H} \varphi(s)$.

(c) $(S, *)$ is *adequate* if and only if $\sigma(H) \neq \emptyset$ for all $H \in \mathcal{P}_f(S)$.

So one can easily see that unlike $(\mathcal{R}, *)$, the partial semigroup $(T, *)$ is adequate. In the case of $(\mathcal{R}, *)$, notice that for any $\mathcal{H} \in \mathcal{P}_f(\mathcal{R})$, $\sigma(\mathcal{H}) \neq \emptyset$ if and only if $\mathcal{H} = \{A :$ A is a matrix with r columns$\}$ for some fixed $r \in \mathbb{N}$.

We are specifically interested in adequate partial semigroups as they lead to an interesting subsemigroup of βS, the Stone-Čech compactification of S. This subsemigroup is itself a compact right topological semigroup and is defined next.

2.3 Definition. Let $(S, *)$ be a partial semigroup. Then

$$\delta S = \bigcap_{x \in S} \overline{\varphi(x)} = \bigcap_{H \in \mathcal{P}_f(S)} \overline{\sigma(H)}$$

Notice that adequacy of S is exactly what is required to guarantee that $\delta S \neq \emptyset$. Also, if S is in fact a semigroup then $\delta S = \beta S$. For an adequate partial semigroup S, δS is in a natural way a compact right topological semigroup.

Being a compact right topological semigroup, δS contains idempotents, left ideals, a smallest 2-sided ideal, and minimal idempotents. Thus δS provides a suitable environment for considering the notion of central. The next example shows that the familiar set $A^\star$ is an adequate partial semigroup also.

2.4 Theorem. *If* $(S,\ \cdot)$ *is an arbitrary semigroup,* $p \in E(\beta S)$*, and* $A \in p$*, then* $(A^\star,\ \cdot)$*, where* $A^\star = \{s \in A :\ s^{-1}A \in p\}$*, is an adequate partial semigroup.*

Proof. Notice that $\cdot$ is a partial operation on $A^\star$, since $A^\star$ fails to be closed under $\cdot$.

We need only show that given any $F \in \mathcal{P}_f(A^\star)$, there exists $a \in A^\star$ such that $Fa \subseteq A^\star$. Since p is an idempotent, $A^\star \in p$. Let $t \in A^\star$. By [Lemma 4.14, 4], $t^{-1}A^\star \in p$. So $A^\star \cap \bigcap_{t \in F} t^{-1}A^\star \neq \emptyset$. Pick $a \in A^\star \cap \bigcap_{t \in A^\star} t^{-1}A^\star$. Then $Fa \subseteq A^\star$ (if $f \in F$, since $a \in f^{-1}A^\star$, we have $fa \in A^\star$). $\square$

3. The Central Sets Theorem for δS

3.1 Definition. Let $(S, *)$ be an adequate partial semigroup and let $A \subseteq S$. Then A is central if and only if there is some minimal idempotent $p \in \delta S$ such that $p \in \overline{A}$.

3.2 Definition. Let $(S, *)$ be an adequate partial semigroup and let $\langle y_n \rangle_{n=1}^{\infty}$ be a sequence in S. Then $\langle y_n \rangle_{n=1}^{\infty}$ is *adequate* if and only if $\prod_{n \in F} y_n$ is defined for each $F \in \mathcal{P}_f(\mathbb{N})$ and for every $K \in \mathcal{P}_f(S)$, there exists $m \in \mathbb{N}$ such that
$FP(\langle y_n \rangle_{n=m}^{\infty}) \subseteq \bigcap_{x \in K} \varphi(x) = \sigma(K)$.

3.3 Definition. Let $(S, *)$ be an adequate partial semigroup. Then S is *commutative* if and only if for all x and $y \in S$, whenever $x \in \varphi(y)$ we have that $y \in \varphi(x)$ and $x * y = y * x$.

3.4 Theorem. *Let S be a commutative adequate partial semigroup, let A be a central subset of S, and for each $l \in \{1,\ 2,\ \ldots,\ k\}$, let $\langle y_{l,n}\rangle_{n=1}^{\infty}$ be an adequate sequence in S. There exist a sequence $\langle a_n\rangle_{n=1}^{\infty}$ in S and a sequence $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that $\max H_n < \min H_{n+1}$ for each $n \in \mathbb{N}$ and such that for each $f : \mathbb{N} \to \{1,\ 2,\ \ldots,\ k\}$, $FP(\langle a_n * \prod_{t\in H_n} y_{f(n),t}\rangle_{n=1}^{\infty}) \subseteq A$.*

Proof. Let p be a minimal idempotent in δS and let $A \in p$. If $A^\star = \{a \in A :\ a^{-1}A \in p\}$, then $A^\star \in p$, and for every $a \in A^\star$, $a^{-1}A = \{b \in \varphi(a) :\ a * b \in A^\star\} \in p$. We claim that given $m,\ r\ \in \mathbb{N}$, $B \in p$ and $F \in \mathcal{P}_f(S)$, there exist $a \in \sigma(F)$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that $\min H > r$, $\prod_{t\in H} y_{l,t} \in \varphi(a)$ and $a * \prod_{t\in H} y_{l,t} \in B$ for every $l \in \{1,\ 2,\ldots,\ m\}$. To see this, let $D = \mathcal{P}_f(S) \times \{r+1,\ r+2,\ldots\}$ be a directed set with ordering defined by $(F_2, n_2) \geq (F_1, n_1)$ if $F_1 \subseteq F_2$ and $n_1 \leq n_2$. For each $(F,n) \in D$ we define $I_{(F,n)} \subseteq S^m$ to be the set of elements of the form $\big(a * \prod_{t\in H} y_{1,t},\ a * \prod_{t\in H} y_{2,t}, \ldots,\ a * \prod_{t\in H} y_{m,t}\big)$, such that $a \in \sigma(F)$ and $H \in \mathcal{P}_f(\mathbb{N})$ satisfies $\min H > n$ and $\prod_{t\in H} y_{l,t} \in \sigma(F * a)$ for every $l \in \{1,\ 2,\ldots,\ m\}$. Notice that $I_{(F,n)} \neq \emptyset$ since $\langle y_{l,n}\rangle$ is an adequate sequence. Define also the set $E_{(F,n)} = I_{(F,n)} \cup \{(a,\ a,\ \ldots,\ a) \in S^m :\ a \in \sigma(F)\}$. Let $(F,n) \in D$. Choose $a \in \sigma(F)$ and $H \in \mathcal{P}_f(\mathbb{N})$ satisfies $\min H > n$ and $\prod_{t\in H} y_{l,t} \in \sigma(F * a)$ for every $l \in \{1,\ 2,\ldots,\ m\}$. Put $\vec{x} = \big(a * \prod_{t\in H} y_{1,t},\ a * \prod_{t\in H} y_{2,t}, \ldots, a * \prod_{t\in H} y_{m,t}\big) \in I_{(F,n)}$ and $\vec{y} = (a,\ a,\ \ldots,\ a) \in S^m$ where $a \in \sigma(F)$, $H \in \mathcal{P}_f(\mathbb{N})$, $\min H > \max(n,r)$ and $\prod_{t\in H} y_{l,t} \in \sigma(a * F)$ for every $l \in \{1,\ 2,\ \ldots,\ m\}$. Choose $G \subseteq \sigma\big(\{a * \prod_{t\in H} y_{1,t},\ a * \prod_{t\in H} y_{2,t}, \ldots, a * \prod_{t\in H} y_{m,t}\}\big)$ and choose $t > \max\{n, \max H\}$. Then we have the following:

(i) $\vec{x} * E_{(G,t)} \subseteq I_{(F,n)}$

(ii) $\vec{y} * E_{(G,t)} \subseteq E_{(F,n)}$

(iii) $\vec{y} * I_{(G,t)} \subseteq I_{(F,n)}$

It follows (as in the proof of Lemma 14.9 in [4], with $T_{(F,n)} = S$ for every $(F,n) \in D$), that $E = \bigcap_{(F,n)\in D} \overline{E}_{(F,n)}$ is a subsemigroup of δS^m and $I = \bigcap_{(F,n)\in D} \overline{I}_{(F,n)}$ is an ideal in E.

Let $\overline{p} = (p,\ p,\ \ldots,\ p) \in \delta S^m$. Then $\overline{p}$ is a minimal idempotent in δS^m. Since $\overline{p} \in E$, $\overline{p}$ is a minimal idempotent in E and hence $\overline{p} \in I$. Now $(\overline{B})^m$ is a neighborhood of $\overline{p}$ in δS^m. So $\overline{p} \in \overline{I}_{(F,n)}$ implies that $(\overline{B})^m \cap I_{(F,n)} \neq \emptyset$, and hence that $B^m \cap I_{(F,n)} \neq \emptyset$. Thus the claim is satisfied.

We now prove the theorem by induction. With $m = m_1,\ r = 1,\ F = \emptyset$ and $B = A^\star$, we can choose $a_1 \in S$ and $H_1 \in \mathcal{P}_f(\mathbb{N})$ such that $\prod_{t\in H_1} y_{l,t} \in \varphi(a_1)$ and $a * \prod_{t\in H_1} y_{l,t} \in A^\star$ for every $l \in \{1,\ 2,\ \ldots, m_1\}$. Assume that a_i and H_i have been defined for every

$i \le n$, with $\max H_i < \min H_{i+1}$ for every $i < n$ and $FP\big(\langle a_i * \prod_{t\in H_i} y_{f(i),t}\rangle_{i=1}^n\big) \subseteq A^\star$ for every $f \in \Phi$. We apply the earlier claim again, with $m = m_n$, $r = \max H_n$, $F = \sigma\Big(\bigcup_{f\in\Phi} FP\big(\langle a_i * \prod_{t\in H_i} y_{f(i),t}\rangle_{i=1}^n\big)\Big)$ and $B = A^\star \cap \bigcap_{f\in\Phi}\big(x^{-1}A^\star :\ x \in FP\big(\langle a_i * \prod_{t\in H_i} y_{f(i),t}\rangle_{i=1}^n\big)\big)$. □

4. Copies of $\mathbb{H}$.

Recall [see 4, Chapter 6] the semigroup $\mathbb{H} = \bigcap_{n\in\mathbb{N}} c\ell_{\beta\mathbb{N}}(2^n\mathbb{N})$. Of particular note is the fact that it is very easy to produce homomorphisms on this semigroup. In fact every finite discrete semigroup is the image of $\mathbb{H}$ under a continuous homomorphism [4, Corollary 6.5]. Motivated by [Theorem 6.15 of 4], the following theorem shows that we can choose our adequate partial semigroup S so that δS is a copy of $\mathbb{H}$.

4.1 Theorem. *Let $\langle A_i\rangle_{i=1}^\infty$ be any sequence of countable sets such that $|A_i| > 1$ for each $i \in \mathbb{N}$. Let e_i be a distinguished element in A_i for each $i \in \mathbb{N}$. Define the set*

$$S = \{\langle a_i\rangle_{i=1}^\infty \in \prod_{i=1}^\infty A_i :\ \{i \in \mathbb{N} :\ a_i \neq e_i\} \text{ is finite }\}$$

If $x = \langle a_i\rangle_{i=1}^\infty \in S$, then $\operatorname{supp}(x) = \{i \in \mathbb{N} :\ a_i \neq e_i\}$. *Let $S^* = \{x \in S :\ \operatorname{supp}(x) \neq \emptyset\}$. For $x = \langle a_i\rangle_{i=1}^\infty$ and $y = \langle b_i\rangle_{i=1}^\infty$ both in S^*, $x * y$ is defined if and only if* $\operatorname{supp}(x) \cap \operatorname{supp}(y) = \emptyset$. *In this case $x * y = \langle c_i\rangle_{i=1}^\infty$ where*

$$c_i = \begin{cases} a_i & \text{if } i \in \operatorname{supp}(x); \\ b_i & \text{if } i \in \operatorname{supp}(y). \end{cases}$$

Then S is an adequate partial semigroup and $\delta S \simeq \mathbb{H}$.

Proof. For each $i \in \mathbb{N}$, choose a group G_i such that $|G_i| = |A_i|$. Let 1_i denote the identity of G_i. Define a bijection $\varphi_i : A_i \to G_i$ such that $\varphi_i(e_i) = 1_i$. Let $\varphi : S \to G$, where $G = \bigoplus_{i\in\mathbb{N}} G_i$, be defined by $\varphi(\langle a_i\rangle_{i=1}^\infty) = \prod_{i=1}^\infty \varphi(a_i)$. Now φ is a bijection. So $\widetilde{\varphi} : \beta S \to \beta G$ is also a bijection [4, Exercise 3.4.1]. If $x,\ y \in S$ and $x * y$ is defined, then $\varphi(x * y) = \varphi(x)\varphi(y)$. Let $p,\ q \in \delta S$. Since

$$\lim_{x\to p}\lim_{y\to q} \varphi(x * y) = \lim_{x\to p}\lim_{y\to q} \varphi(x)\varphi(y),$$

we have $\widetilde{\varphi}(p * q) = \widetilde{\varphi}(p)\widetilde{\varphi}(q)$. So $\widetilde{\varphi}$ is a homomorphism on δS.

Now let $U_i = \{a \in G :\ \pi_j(a) = 1_j$ whenever $j \le i\}$. We claim that $\widetilde{\varphi}[\delta S] = \bigcap_{i\in\mathbb{N}} c\ell_{\beta G}(U_i)\backslash\{1\}$. To see this, let $V_i = \{x \in S : \min\big(\operatorname{supp}(x)\big) \ge i\}$ for each $i \in \mathbb{N}$. Notice that $\delta S = \bigcap_{i\in\mathbb{N}} c\ell_{\beta S}(V_i)\backslash\{u\}$ (where u is the unique member of S with empty support). Now $\varphi[V_i\backslash\{u\}] = U_i\backslash\{1\}$, where 1 is the identity in G.

Hence $\widetilde{\varphi}[c\ell_{\beta S}(V_i)\backslash\{u\}] = c\ell_{\beta G}(U_i)\backslash\{1\}$. So $\widetilde{\varphi}\big[\bigcap_{i=1}^{\infty}\big(c\ell_{\beta S}(V_i)\backslash\{u\}\big)\big] = \bigcap c\ell_{\beta G}(U_i)\backslash\{1\}$. Therefore
$\bigcap_{i=1}^{\infty}\widetilde{\varphi}[c\ell_{\beta S}V_i\backslash\{u\}] = \bigcap_{i=1}^{\infty} c\ell_{\beta G}U_i\backslash\{1\}$. We have [Exercise 7.2.5, 4] that
$\bigcap_{i=1}^{\infty} c\ell_{\beta G}U_i\backslash\{1\} \simeq \mathbb{H}$. Since $\bigcap_{i=1}^{\infty}\widetilde{\varphi}[c\ell_{\beta S}V_i\backslash\{u\}] = \delta S$, $\delta S \simeq \mathbb{H}$. □

Recall that a sequence $\langle x_n\rangle_{n=1}^{\infty}$ is adequate if and only if $\prod_{n\in F} x_n$ is defined for each $F \in \mathcal{P}_f(\mathbb{N})$ and for every $K \in \mathcal{P}_f(S)$, there exists $m \in \mathbb{N}$ such that $FP(\langle x_n\rangle_{n=m}^{\infty}) \subseteq \cap_{y\in K}\varphi(y)$.

4.2 Lemma. *Let S be an adequate partial semigroup and let $\langle x_n\rangle_{n=1}^{\infty}$ be an adequate sequence in S and let $T = \bigcap_{m=1}^{\infty}\overline{FP(\langle x_n\rangle_{n=m}^{\infty})}$. Then T is a subsemigroup of δS.*

Proof. From the definition of adequate sequence, we see immediately that $T \subseteq \delta S$. To see that T is a semigroup, let $p, q \in T$. For $m \in \mathbb{N}$, let $T_m = FP(\langle x_n\rangle_{n=m}^{\infty})$. To see that $p * q \in T$, let $m \in \mathbb{N}$. We shall show that $T_m \in p * q$. Given $y \in T_m$, pick $F \in \mathcal{P}_f(\mathbb{N})$ such that $y = \prod_{n\in F} x_n$ and $\min F \geq m$. Let $k = \max F + 1$. Then $y * T_k \subseteq T_m$ so $T_m \in p * q$ as required. □

Theorem 4.4 below is the extension of Theorem 6.27 [4] to adequate partial semigroups, giving us a copy of $\mathbb{H}$ in δS.

4.3 Definition. Given a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in some semigroup S, we say that $\langle x_n\rangle_{n=1}^{\infty}$ has distinct finite products if whenever F and G are distinct members of $\mathcal{P}_f(\mathbb{N})$, then $\prod_{n\in F} x_n \neq \prod_{n\in G} x_n$.

4.4 Theorem. *Let S be a discrete adequate partial semigroup and let $\langle x_n\rangle_{n=1}^{\infty}$ be an adequate sequence in S with distinct finite products. Let $T = \bigcap_{m=1}^{\infty} c\ell\big(FP(\langle x_m\rangle_{n=m}^{\infty})\big)$. Then T is a subsemigroup of δS which is algebraically and topologically isomorphic to $\mathbb{H}$.*

Proof. By Lemma 4.2 we have that T is a semigroup of δS. We define a mapping $f : FP(\langle x_m\rangle_{n=1}^{\infty}) \to 2\mathbb{N}$ by stating that $f(\prod_{n\in F} x_n) = \sum_{n\in F} 2^n$ for every $F \in \mathcal{P}_f(\mathbb{N})$. Since f is a bijection $\widetilde{f} : \overline{FP(\langle x_n\rangle_{n=1}^{\infty})} \to \overline{2\mathbb{N}}$ is a homeomorphism by [5, Exercise 3.4.1].

As above, for each $m \in \mathbb{N}$, let $T_m = FP(\langle x_n\rangle_{n=m}^{\infty})$. Then $f[T_m] = 2^m\mathbb{N}$ and so

$$\widetilde{f}[T] = \widetilde{f}[\textstyle\bigcap_{m=1}^{\infty}\overline{T_m}\,] = \bigcap_{m=1}^{\infty}\widetilde{f}[\overline{T_m}\,] = \bigcap_{m=1}^{\infty}\overline{f[T_m]\,} = \bigcap_{m=1}^{\infty}\overline{2^m\mathbb{N}} = \mathbb{H}.$$

To see that $\widetilde{f}_{|T}$ is a homomorphism, let $p, q \in T$. To see that $\widetilde{f}(p*q) = \widetilde{f}(p) + \widetilde{f}(q)$, it suffices to show that the continuous functions $\widetilde{f}\circ\rho_p$ and $\rho_{\widetilde{f}(q)}\circ\widetilde{f}$ agree on T_1, which is a member of p. So let $y \in T_1$ and pick $F \in \mathcal{P}_f(\mathbb{N})$ such that $y = \prod_{n \in F} x_n$. Let $m = \max F + 1$. To see that $\widetilde{f}(y * q) = f(y) + \widetilde{f}(q)$, it suffices to observe that the continuous functions $\widetilde{f} \circ \lambda_y$ and $\lambda_{f(y)} \circ \widetilde{f}$ agree on T_m, a member of q. $\square$

We now show that one can, under certain conditions, guarantee the existence of adequate sequences with distinct finite products.

4.5 Definition. Let S be an adequate partial semigroup.

i) S is *right cancellative* if for all y and $z \in S$ and for all $x \in \varphi(y) \cap \varphi(z)$, whenever $yx = zx$, then $y = z$.

ii) S is *weakly left cancellative* if for all a, and $b \in S$, $\{x \in S : x \in \varphi(a) \text{ and } ax = b\}$ is finite.

4.6 Definition. A partial semigroup S is said to be *strongly adequate* if and only if for all $F \in \mathcal{P}_f(S)$, $\sigma(F)$ is infinite.

Note that if S is not strongly adequate then δS is a subsemigroup (a subset which is a semigroup) of S.

4.7 Theorem. *If S is a countable cancellative strongly adequate partial semigroup then S contains an adequate sequence with distinct finite products.*

Proof. Enumerate S as $\langle s_n \rangle_{n=1}^{\infty}$. Choose $y \in S$. Inductively let $n \in \mathbb{N}$ and assume we have chosen a sequence $\langle y_t \rangle_{t=1}^{n}$ satisfying:

i) uniqueness of finite products,

ii) $\prod_{t \in F} y_t$ is defined for each $F \in \mathcal{P}_f(\{1, 2, \ldots, n\})$, and

iii) for all $k \in \{1, 2, \ldots, n\}$, and for all $\emptyset \neq F \subseteq \{k, k+1, \ldots, n\}$, we have that $\prod_{t \in F} y_t \in \sigma(\{s_1, s_2, \ldots, s_k\})$.

Define the set

$$A = \Big\{\sigma\big(\{s_1 a, s_2 a \ldots, s_k a\}\big) : k \in \{1, 2, \ldots, n\}, \emptyset \neq F \subseteq \{k, \ldots, n\}, \text{ and } a = \prod_{t \in F} y_t\Big\};$$

and let $B = A \cap \bigcap \sigma\big(\{s_1, s_2, \ldots, s_{n+1}\}\big) \cap \bigcap \Big\{\varphi\big(\prod_{t \in F} y_t\big) : \emptyset \neq F \subseteq \{1, 2, \ldots, n\}\Big\}$. Then B is infinite since S is strongly adequate.

Define sets A_1, A_2, and A_3 as follows:

$$A_1 = \Big\{z \in S : \exists\, F \subseteq \mathcal{P}_f(\{1, 2, \ldots, n\}) \text{ such that } z = \prod_{t \in F} y_t\Big\},$$

$$A_2 = \Big\{z \in S : \exists\, F \neq G \subseteq \mathcal{P}_f(\{1, 2, \ldots, n\}) \text{ such that } \prod_{t \in F} y_t \cdot z = \prod_{t \in G} y_t \cdot z\Big\},$$

$$A_3 = \Big\{z \in S : \exists\, F, G \subseteq \mathcal{P}_f(\{1, 2, \ldots, n\}) \text{ such that } \prod_{t \in F} y_t = \prod_{t \in G} y_t \cdot z\Big\}.$$

Let $C = A_1 \cup A_2 \cup A_3$. Then C is finite since S is right cancellative and weakly left cancellative.

Now pick $y_{n+1} \in B \backslash C$. $\square$

References

[1] V. Bergelson, A. Blass, and N. Hindman, *Partition theorems for spaces of variable words*, Proc. London Math. Soc. **68** (1994), 449-476.

[2] V. Bergelson and N. Hindman, *Nonmetrizable topological dynamics and Ramsey Theory*, Trans. Amer. Math. Soc. **320** (1990), 293-320.

[3] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton University Press, Princeton, 1981.

[4] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification – theory and applications*, W. de Gruyter & Co., Berlin, 1998.

[5] J. McLeod, *Some Notions of size in partial semigroups*, Topology Proc. **25** (2000), 317-332.

[6] J. McLeod, *Notions of size in adequate partial semigroups*, Ph.D thesis, Howard University, 2001.

[7] H. Shui and H. Yang, *Nonmetrizable topological dynamic characterization of central sets*, Fundamenta Math. **150** (1996), 1-9.

Jillian McLeod
Department of Mathematics and Statistics
Mount Holyoke College
South Hadley, MA 01075
`jmcleod@mtholyoke.edu`