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Notions of Size in Adequate Partial Semigroups

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DEDICATION

To those who came before and those who will come after.

To my parents Errol & Gillian McLeod,

my sister Nikkie,

and my brothers Kurt, Ian, and Jason.

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This has been a long road; and I was carried along it on the wings of many angels. I hope that this work is a worthy expression of gratitude to my Creator. Thank you from whom all knowledge flows.

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"One must also accept that one has 'uncreative' moments.

The more honestly one can accept that,

the quicker these moments will pass.

One must have the courage to call a halt,

to feel empty and discouraged."

Etty Hillesum

ABSTRACT

Often it is necessary to restrict the natural operation on a given set S to a subset D of S on which the operation is well behaved. In the literature, terms such as *groupoid, pargoid*, and *oid* have been used to describe structures having these partial operations. A *partial semigroup* is one such structure. Loosely defined, a partial semigroup is a pair $(S, *)$, such that S is a nonempty set and $*$ is a binary operation which is associative where it is defined.

Introduced in [1], partial semigroups allow us to explore partial operations in a given semigroup. Specifically, we consider how notions of size in a semigroup (with its total operation), are affected when extended to a partial semigroup (with its restricted operation). It is very often difficult, if not impossible, to make direct transfers of results from total operations to partial operations. Thus, it is often necessary that we modify the "usual" definitions, characterizations, or axioms, to extend to the partial case. These, sometimes minor, modifications can unveil new and interesting questions that are specific to the partial operation.

Our particular interest in partial operations is in the context of the compact right topological semigroup βS. Much is known about the algebra of βS, the Stone-Čech compactification of any semigroup S. Partial semigroups give rise to a natural and useful subset of βS which is in fact a *semigroup.* This subset is δS. In [1] it was shown that δS is a compact Hausdorff right topological semigroup. Therefore δS is guaranteed the structure common to any such object; and we are therefore able to, in many cases, easily extend our knowledge of operations in semigroups to partial operations in adequate partial semigroups.

Borrowing from the field of Topological Dynamics, the notions of size we discuss are *thick, syndetic, piecewise syndetic, IP*, and *central.* These notions have all been useful in the development of the theory of βS. For a semigroup S, it is known that many of these notions of size can be characterized in terms of $K(\beta S)$,

the smallest ideal of βS, [9]. We show here that each notion has natural extensions, combinatorially and algebraically, to partial semigroups. And we answer, for each notion, the following question: *Does the equivalence, that holds in a semigroup, between the combinatorial and algebraic characterization of the given notion, transfer to adequate partial semigroups?*

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CHAPTER I

PRELIMINARIES

The concept of a *partial semigroup* arose in the study of partition theorems for variable words. In their paper entitled "Partition theorems for spaces of variable words," [1], the authors noticed that the natural binary operation on located words is well defined only when it is restricted to located words with disjoint unions. They also noticed that the natural operation does not allow the operation of concatenation to be a homomorphism. These problems are easily remedied with the introduction of the notion of partial semigroups, which allows one to restrict the natural operation to the subset for which it is defined or works "well".

In this chapter we develop the notion of a partial semigroup and provide the necessary algebra for our research. We see that partial semigroups arise essentially in one of the two ways referred to above. Partial semigroups, in particular *adequate partial semigroups*, lead to a natural and useful subset of βS, the Stone-Čech compactification of S, which is in fact a semigroup. This subset, δS, will provide the algebraic foundation for all that we do in the subsequent chapters. Being a subset of βS, the algebra of δS will be developed from much of what we already know about βS. Therefore we will provide many results, all previously known, about the algebraic and combinatorial structure of βS and then attempt to extend them to partial semigroups.

Our interest in partial semigroups comes at a time when there is a growing interest in, and possibly need for, the study of partial operations as a separate area of algebra [10]. Partial operations are frequently investigated in connection with the areas of information theory, category theory, and coding theory. Here we investigate partial operations in the context of "large" sets. Some important notions of largeness in a semigroup S can be characterized in terms of $K(\beta S)$, the smallest ideal of βS. It is our intention to use $K(\delta S)$, the smallest ideal of δS, to obtain appropriate analogues of these notions for partial semigroups.

The questions we address for partial semigroups are generally in an attempt to transfer our knowledge about notions of size in a semigroup S. Theorem 1.9 below, shows that for an adequate partial semigroup S, δS is in a natural way, a compact Hausdorff right topological semigroup. This fact allows us, in many cases, to easily extend our knowledge of total operations in semigroups to partial operations in adequate partial semigroups.

There are many notions of size in an arbitrary semigroup. Many of these notions originate in topological dynamics in the context of the semigroup $(\mathbb{N}, +)$. These notions, such as *central*, have been very important in excavating the rich structure of βS. In a semigroup, it has been shown that the combinatorial definitions of these notions of size, all have nice algebraic characterizations in βS [9]. In extending the notions: *thick, syndetic, piecewise syndetic, IP, and central*, to a partial semigroup we, in general, see a loss of this equivalence between the combinatorial definitions and the algebraic characterization. In some cases we are able to find conditions under which equivalence may be attained. In most cases we simply produce new equivalences.

We shall be using the Stone- Čech compactification, βS, of a discrete space S. We take the points of βS to be the ultrafilters on S; the principal ultrafilters being identified with the points of S. By this identification, we pretend that $S \subseteq \beta S$. In a similar fashion, if $S \subseteq T$, we pretend that $\beta S \subseteq \beta T$ by identifying the ultrafilter p on S with the ultrafilter $\{A \subseteq T : A \cap S \in p\}$ on T. Given a set $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (as well as a basis for the closed sets) of βS.

If S is a semigroup, there is a natural extension of the operation $\cdot$ of S to βS making βS a compact right topological semigroup with S contained in its topological center. This says that for each $p \in \beta S$ the function $\rho_p : \beta S \to \beta S$, defined by $\rho_p(q) = q \cdot p$, is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$, defined by $\lambda_x(q) = x \cdot q$ is continuous.

Any compact Hausdorff right topological semigroup $(T, \cdot)$ has a smallest two sided ideal $K(T)$ which is the union of all of the minimal left ideals of T, each of which is closed [9, Theorem 2.8 and Corollary 2.6], and any compact right topological semigroup contains idempotents [9, Theorem 2.5]. Since the minimal left ideals are themselves compact right topological semigroups, this says in particular that there are idempotents in the smallest ideal. There is a partial ordering of the idempotents of T determined by $p \leq q$ if and only if $p = p \cdot q = q \cdot p$. An idempotent p is minimal with respect to this order if and only if $p \in K(T)$ [9, Theorem 1.59].

This ends our brief review of the algebra of βS. We shall frequently cite [9] for other basic properties of βS when needed. We begin now our introduction to partial semigroups. Much of the material overlaps with that in [5] and [8]. The two examples that follow indicate how we often in fact encounter partial semigroups.

Given a set S, and a natural binary operation, it is often convenient to define the operation for only a subset of $S \times S$. Consider for instance the semigroup $(\mathcal{P}_f(\mathbb{N}), \cup)$, where $\mathcal{P}_f(\mathbb{N}) = \{F : F \text{ is a finite nonempty subset of } \mathbb{N}\}$. If we define $\varphi : (\mathcal{P}_f(\mathbb{N}), \cup) \to (\mathbb{N}, +)$ by $\varphi(F) = |F|$, then φ is not a homomorphism. However, if we let

$$A \uplus B = \begin{cases} A \cup B & \text{if } A \cap B = \emptyset \\ \text{undefined} & \text{if } A \cap B \neq \emptyset \end{cases}$$

then φ is a homomorphism on $(\mathcal{P}_f(\mathbb{N}), \uplus)$, in the sense that $\varphi(A \uplus B) = \varphi(A) + \varphi(B)$ whenever $A \uplus B$ is defined.

Another case in which we may need to restrict the domain of the operation occurs when the natural operation does not satisfy the closure property. For example, given a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in the semigroup $(S, \cdot)$, let

$$T = FP(\langle x_n \rangle_{n=1}^{\infty}) = \{\prod_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\},$$

where the products are taken in increasing order of indices. Then $(x_1 \cdot x_3) \cdot (x_2 \cdot x_4)$ is not likely to be in T unless x_2 and x_3 commute, and $(x_1 \cdot x_3) \cdot (x_3 \cdot x_4)$ is not likely to be in T at all. On the other hand, if we let $(\prod_{n \in F} x_n) * (\prod_{n \in G} x_n)$ be

$$\begin{cases} \prod_{n \in F \cup G} x_n & \text{if } \max F < \min G \\ \text{undefined} & \text{if } \max F \geq \min G \end{cases}$$

Then T is closed under $*$.

$(\mathcal{P}_f(\mathbb{N}), \uplus)$ and $(T, *)$ are examples of adequate partial semigroups.

1.1 Definition. A *partial semigroup* is a pair $(S, *)$, where S is a nonempty set and $*$ maps a subset D of $S \times S$ to S so that for all $x, y, z \in S$,

(a) if $(x, y) \in D$ and $(x * y, z) \in D$, then $(y, z) \in D$, $(x, y * z) \in D$, and $(x * y) * z = x * (y * z)$ and

(b) if $(y, z) \in D$ and $(x, y * z) \in D$, then $(x, y) \in D$, $(x * y, z) \in D$, and $(x * y) * z = x * (y * z)$.

If $(S, *)$ is a partial semigroup and $(x, y) \in \text{domain}(*)$, we say that "$x * y$ is defined". The requirements of Definition 1.1(a) and (b), can then be more succinctly stated as "$(x * y) * z = x * (y * z)$ in the sense that, whenever either side is defined, so is the other and they are equal."

1.2 Definition. Let $(S, *)$ be a partial semigroup.

(a) For $x \in S$, $\varphi(x) = \varphi_S(x) = \{y \in S : x * y \text{ is defined}\}$.

(b) For each $H \in \mathcal{P}_f(S)$, $\sigma(H) = \bigcap_{s \in H} \varphi(s)$.

(c) $\sigma(\emptyset) = S$.

(d) The semigroup S is *adequate* if and only if $\sigma(H) \neq \emptyset$ for all $H \in \mathcal{P}_f(S)$.

(e) $\delta S = \bigcap_{x \in S} c\ell_{\beta S}(\varphi(x)) = \bigcap_{H \in \mathcal{P}_f(S)} c\ell_{\beta S}(\sigma(H))$.

All of the partial semigroups that we have presented thus far have been adequate. The assertion that S is adequate is exactly the assertion that $\delta S \neq \emptyset$. Notice that any semigroup S is a partial semigroup in which case $\delta S = \beta S$. Also note that the operation $*$ of a partial semigroup S is defined precisely on $\bigcup_{x \in S} \big(\{x\} \times \varphi(x)\big)$.

A fundamental fact for our discussion of notions of size in partial semigroups is that, for an adequate partial semigroup S, δS is in a natural way a compact right topological *semigroup*.

1.3 Theorem. *Let $(S,*)$ be an adequate partial semigroup. Let*

$$D = \left(\bigcup_{x \in S} (\{x\} \times \overline{\varphi(x)}\right) \cup (\beta S \times \delta S)\,.$$

Then the operation $*$ can be extended uniquely to D so that

(a) for each $x \in S$, the function $\lambda_x : \overline{\varphi(x)} \to \beta S$, defined by $\lambda_x(q) = x * q$, is continuous, and

(b) for each $p \in \delta S$, the function $\rho_p : \beta S \to \beta S$, defined by $\rho_p(q) = q * p$ is continuous.

Proof. For each $x \in S$, define $l_x : \varphi(x) \to S$ by $l_x(y) = x * y$. Then l_x has a unique continuous extension $\widetilde{l_x} : \overline{\varphi(x)} \to \beta S$. For $q \in \beta S$, define $x * q = \widetilde{l_x}(q)$ whenever $x * q$ has not already been defined. Then λ_x is continuous.

Now, for each $p \in \delta S$, $x * p$ is defined for all $x \in S$. Define $r_p(x) = x * p$ and let $\widetilde{r_p} : \beta S \to \beta S$ be the unique continuous extension of r_p. For $q \in \beta S$, define $q * p = \widetilde{r_p}(q)$ whenever $q * p$ has not already been defined. $\square$

As a subset of βS, the points of δS are ultrafilters. Thus, we are interested in describing the members of $p * q$ in terms of the ultrafilters p and q in δS.

1.4 Definition. Let $(S,*)$ be a partial semigroup, let $x \in S$, and let $A \subseteq S$. Then $x^{-1}A = \{y \in \varphi(x) : x * y \in A\}$.

Note that, as in a semigroup, and even more strongly here, there is no suggestion, even in the event that S has an identity, that any or all elements of S have inverses. Also, if the operation in S is denoted by $+$, then we write $-x+A$ for $\{y\in\varphi(x): x+y\in A\}$.

1.5 Lemma. *Let $(S,*)$ be a partial semigroup, let $A\subseteq S$, and let $a,b,c\in S$. Then*

$$c\in b^{-1}(a^{-1}A)\Leftrightarrow b\in\varphi(a) \text{ and } c\in(a*b)^{-1}A.$$

In particular, if $b\in\varphi(a)$, then $b^{-1}(a^{-1}A)=(a*b)^{-1}A$.

Proof.

$$\begin{aligned} c\in b^{-1}(a^{-1}A) &\Leftrightarrow c\in\varphi(b) \text{ and } b*c\in a^{-1}A\\ &\Leftrightarrow c\in\varphi(b) \text{ and } b*c\in\varphi(a) \text{ and } a*(b*c)\in A\\ &\Leftrightarrow b\in\varphi(a) \text{ and } c\in\varphi(a*b) \text{ and } (a*b)*c\in A\\ &\Leftrightarrow b\in\varphi(a) \text{ and } c\in(a*b)^{-1}A \end{aligned}$$

□

For any semigroup S and any member x of S, we are guaranteed that $S\backslash x^{-1}(S\backslash A)=x^{-1}A$. This need not be the case in a partial semigroup.

1.6 Lemma. *Let $(S,*)$ be a partial semigroup. Let $x\in S$ and $A\subseteq S$. Then $S\backslash x^{-1}(S\backslash A)=\big(S\backslash\varphi(x)\big)\cup(x^{-1}A)$.*

Proof. Let $y\in S\backslash x^{-1}(S\backslash A)$. Then either $y\notin\varphi(x)$ or $y\in\varphi(x)$ and $x*y\notin S\backslash A$. So $S\backslash x^{-1}(S\backslash A)\subseteq\big(S\backslash\varphi(x)\big)\cup(x^{-1}A)$.

Now if $y\in x^{-1}A$, then $y\in\varphi(x)$ and $x*y\in A$. So $y\in S$ and $x*y\notin S\backslash A$. So $y\in S\backslash(x^{-1}(S\backslash A))$. On the other hand, if $y\in\big(S\backslash\varphi(x)\big)$, then $y\notin x^{-1}(S\backslash A)$. □

1.7 Lemma. *Let S be an adequate partial semigroup.*

(a) Let $x\in S$, let $q\in\overline{\varphi(x)}$, and let $A\subseteq S$. Then $A\in x*q$ if and only if $x^{-1}A\in q$.

(b) Let $p\in\beta S$, let $q\in\delta S$, and let $A\subseteq S$. Then $A\in p*q$ if and only if $\{x\in S: x^{-1}A\in q\}\in p$.

Proof. (a) Necessity. Pick $B \in q$ such that $\lambda_x\left[\overline{B \cap \varphi(x)}\right] \subseteq \overline{A}$. Then $\varphi(x) \cap B \subseteq x^{-1}A$.

Sufficiency. Suppose that $A \notin x * q$. Then $S \backslash A \in x * q$ so that, by the already established necessity, $x^{-1}(S \backslash A) \in q$ while $x^{-1}A \cap x^{-1}(S \backslash A) = \emptyset$, a contradiction.

(b) Necessity. Pick $B \in p$ such that $\rho_q\left[\overline{B}\right] \subseteq \overline{A}$. Then by (a), $B \subseteq \{x \in S : x^{-1}A \in q\}$.

Sufficiency. Suppose that $A \notin p * q$. Then $S \backslash A \in p * q$ so that, by the already established necessity, $\{x \in S : x^{-1}(S \backslash A) \in q\} \in p$ while $\{x \in S : x^{-1}(S \backslash A) \in q\} \cap \{x \in S : x^{-1}A \in q\} = \emptyset$, a contradiction. □

1.8 Lemma. *Let S be an adequate partial semigroup, let $p \in \beta S$, $q \in \delta S$, and $a \in S$. Then $\varphi(a) \in p * q$ if and only if $\varphi(a) \in p$.*

Proof. Necessity. Assume that $\varphi(a) \in p * q$ so that $\{b \in S : b^{-1}\varphi(a) \in q\} \in p$. We show that $\{b \in S : b^{-1}\varphi(a) \in q\} \subseteq \varphi(a)$. So let $b^{-1}\varphi(a) \in q$. Pick $c \in b^{-1}\varphi(a)$. Then $c \in \varphi(b)$ and $b * c \in \varphi(a)$ so $a * (b * c)$ is defined and thus $a * (b * c) = (a * b) * c$ and in particular $b \in \varphi(a)$.

Sufficiency. Assume that $\varphi(a) \in p$. We claim that $\varphi(a) \subseteq \{b \in S : b^{-1}\varphi(a) \in q\}$ so that $\varphi(a) \in p * q$. Let $b \in \varphi(a)$. Since $q \in \delta S$, $\varphi(a * b) \in q$. Therefore it suffices to show that $\varphi(a * b) \subseteq b^{-1}\varphi(a)$. Let $c \in \varphi(a * b)$. Then $(a * b) * c = a * (b * c)$ so $c \in \varphi(b)$ and $b * c \in \varphi(a)$. That is, $c \in b^{-1}\varphi(a)$ as required. □

1.9 Theorem. *Let S be an adequate partial semigroup. Then with the restriction of the operation given in Theorem* 1.3, *δS is a compact right topological semigroup.*

Proof. We have by Lemma 1.8 that if $p, q \in \delta S$, then $p * q \in \delta S$. Since δS is a closed subset of βS we have that δS is compact. By Theorem 1.3, we have that ρ_q is continuous for each $q \in \delta S$. It thus suffices to show that the operation is associative on δS.

To this end, let $p, q, r \in \delta S$. Suppose that $p * (q * r) \neq (p * q) * r$ and pick

$A \in p * (q * r) \backslash (p * q) * r$. Let $B = \{a \in S : a^{-1}(S \backslash A) \in r\}$. Then $B \in p * q$ so $\{b \in S : b^{-1}B \in q\} \in p$. Also, $\{b \in S : b^{-1}A \in q * r\} \in p$ so pick $b \in S$ such that $b^{-1}B \in q$ and $b^{-1}A \in q * r$. Then $\{c \in S : c^{-1}(b^{-1}A) \in r\} \in q$ so pick $c \in b^{-1}B$ such that $c^{-1}(b^{-1}A) \in r$. Then $c \in \varphi(b)$ and $b * c \in B$ so $(b * c)^{-1}(S \backslash A) \in r$. Pick $a \in c^{-1}(b^{-1}A) \cap (b * c)^{-1}(S \backslash A)$. Then $a \in \varphi(c)$ and $c * a \in b^{-1}A$ so $c * a \in \varphi(b)$ and $b * (c * a) \in A$. On the other hand, $a \in \varphi(b * c)$ and $(b * c) * a \in S \backslash A$, a contradiction.

□

As noted earlier, the fact of Theorem 1.9 allows for the algebraic part of our investigation of the notions *thick, syndetic* (Chapter 2), *piecewise syndetic* (Chapter 3), *IP* and *central* (Chapter 4). As a compact right topological semigroup, δS enjoys all of the properties characteristic of those structures. That is, δS contains idempotents, left ideals, minimal left ideals, minimal idempotents, and a smallest ideal $K(\delta S)$.

It would seem natural to define an "adequate partial subsemigroup" S of an adequate semigroup T to be a subset which is an adequate partial semigroup under the inherited operation. We see now that this is not enough to guarantee that $\delta S \subseteq \delta T$.

1.10 Lemma. *Let T be an adequate partial semigroup and let S be a subset of T which is an adequate partial semigroup under the inherited operation. Then the following statements are equivalent.*

(a) $\delta S \subseteq \delta T$.

(b) For all $y \in T$ there exists $H \in \mathcal{P}_f(S)$ such that $\bigcap_{x \in H} \varphi_S(x) \subseteq \varphi_T(y)$.

(c) For all $F \in \mathcal{P}_f(T)$ there exists $H \in \mathcal{P}_f(S)$ such that
$\bigcap_{x \in H} \varphi_S(x) \subseteq \bigcap_{x \in F} \varphi_T(x)$.

Proof. (a) $\Rightarrow$ (b). Let $y \in T$ and suppose that for all $H \in \mathcal{P}_f(S)$,

$$\bigcap_{x \in H} \varphi_S(x) \backslash \varphi_T(y) \neq \emptyset\,.$$

Then $\{\bigcap_{x\in H} \varphi_S(x)\backslash\varphi_T(y) : H \in \mathcal{P}_f(S)\}$ has the finite intersection property so pick $p \in \beta S$ such that $\{\bigcap_{x\in H} \varphi_S(x)\backslash\varphi_T(y) : H \in \mathcal{P}_f(S)\} \subseteq p$. Then $p \in \delta S\backslash\delta T$, a contradiction.

The other implications are immediate □

We shall see in Theorem 1.13 that the condition of Lemma 1.10 does not have to hold.

1.11 Definition. Let T be a partial semigroup. Then S is an *adequate partial subsemigroup* of T if and only if $S \subseteq T$, S is an adequate partial semigroup under the inherited operation, and for all $F \in \mathcal{P}_f(T)$ there exists $H \in \mathcal{P}_f(S)$ such that $\bigcap_{x\in H} \varphi_S(x) \subseteq \bigcap_{x\in F} \varphi_T(x)$.

1.12 Remark. *Notice that:*

(1) By Lemma 1.10 *if S is an adequate partial subsemigroup of T, then $\delta S \subseteq \delta T$.*

(2) If T is a partial semigroup which has an adequate partial subsemigroup, then necessarily T is an adequate partial semigroup.

(3) If S is a subset of T which is a partial semigroup under the inherited operation, then every adequate partial subsemigroup of T included in S is an adequate partial subsemigroup of S.

Notice that "is an adequate partial subsemigroup of" is a transitive relation. However, the following result establishes that the notion is not as well behaved as one might like.

1.13 Theorem. *There exist an adequate partial semigroup T and adequate partial subsemigroups R and S of T such that $R \cap S$ is an adequate partial semigroup with the inherited operation, but $R \cap S$ is not an adequate partial subsemigroup of T.*

Proof. Let $T = \mathcal{P}_f(\omega + \omega)$, where $\omega + \omega$ is the ordinal sum. For $\alpha, \beta \in T$, define $\alpha * \beta = \alpha \cup \beta$ exactly when $\max \alpha < \min \beta$. It is easy to see that T is an adequate partial semigroup.

Let $A = \omega \cup \{\omega + 2n : n \in \omega\}$ and $B = \omega \cup \{\omega + 2n + 1 : n \in \omega\}$. Let $R = \mathcal{P}_f(A)$ and let $S = \mathcal{P}_f(B)$. It is routine to verify that both R and S are adequate partial subsemigroups of T. Now $R \cap S = \mathcal{P}_f(\omega)$. To see that $R \cap S$ is not an adequate partial subsemigroup of T, let $F = \big\{\{\omega\}\big\}$. Then there is no $H \in \mathcal{P}_f(R \cap S)$ such that $\bigcap_{\alpha \in H} \varphi_{R \cap S}(\alpha) \subseteq \bigcap_{\alpha \in F} \varphi_T(\alpha)$ (which is $\{\alpha \in T : \min \alpha > \omega\}$). $\square$

Theorem 1.13 shows in particular that one may have adequate partial semigroups S and T such that $S \subseteq T$ (and S has the inherited operation) but $\delta S \backslash \delta T \neq \emptyset$. If $q \in \delta S \backslash \delta T$ and $p \in \beta S \backslash S$, then $p * q$ is defined in βS, but is not defined in βT. This fact raises the possibility of some ambiguity concerning what is meant by $p * q$. The following result shows that, if it is defined, $p * q$ can mean only one thing.

1.14 Lemma. *Let T be an adequate partial semigroup and let R and S be subsets of T which are both adequate partial semigroups under the inherited operation. Let $p, q \in \beta(R \cap S)$. If $p * q$ is defined in βS and $p * q$ is defined in βR, then it is the same object under both definitions.*

Proof. Let $A \subseteq R \cap S$ and assume that $A \in p * q$ as that object is defined in βR. We show that $A \in p * q$ as that object is defined in βS. Assume first that $p \in R \cap S$ so that (because $p * q$ is defined), $\varphi_R(p) \in q$ and $\varphi_S(p) \in q$. Then by Lemma 1.7(a) $\{y \in \varphi_R(p) : p * y \in A\} \in q$ and $\{y \in \varphi_R(p) : p * y \in A\} \cap \varphi_S(p) \subseteq \{y \in \varphi_S(p) : p * y \in A\}$ and hence $\{y \in \varphi_S(p) : p * y \in A\} \in q$.

Now assume that $p \in \beta(R \cap S) \backslash (R \cap S)$ and hence (because $p * q$ is defined), $q \in (\delta R \cap \delta S)$. Then by Lemma 1.7(b) $\{x \in R : \{y \in \varphi_R(x) : x * y \in A\} \in q\} \in p$. Also $S \in p$. We claim that

$$\{x \in R : \{y \in \varphi_R(x) : x * y \in A\} \in q\} \cap S \subseteq \{x \in S : \{y \in \varphi_S(x) : x * y \in A\} \in q\}$$

so that $\{x \in S : \{y \in \varphi_S(x) : x * y \in A\} \in q\} \in p$ as required. To this end, let $x \in S \cap R$ such that $\{y \in \varphi_R(x) : x * y \in A\} \in q$. Since also $\varphi_S(x) \in q$ and

$$\{y \in \varphi_R(x) : x * y \in A\} \cap \varphi_S(x) \subseteq \{y \in \varphi_S(x) : x * y \in A\}$$

we have that $\{y \in \varphi_S(x) : x * y \in A\} \in q$ as required. $\square$

Next we define homomorphisms between partial semigroups and establish conditions guaranteeing that the continuous extension of a partial semigroup homomorphism is a homomorphism.

1.15 Definition. Let S and T be partial semigroups and let $f : S \to T$. Then f is a *partial semigroup homomorphism* if and only if whenever $x \in S$ and $y \in \varphi_S(x)$, one has that $f(y) \in \varphi_T\big(f(x)\big)$ and $f(x * y) = f(x) * f(y)$.

1.16 Lemma. *Let S and T be adequate partial semigroups, let $f : S \to T$ be a partial semigroup homomorphism, and let $\widetilde{f} : \beta S \to \beta T$ be the continuous extension of f. If $p \in \beta S$, $q \in \delta S$, and $\widetilde{f}(q) \in \delta T$, then $\widetilde{f}(p * q) = \widetilde{f}(p) * \widetilde{f}(q)$. If $f[S]$ is an adequate partial subsemigroup of T, then $\widetilde{f}[\delta S] \subseteq \delta T$ and $\widetilde{f}_{|\delta S}$ is a (semigroup) homomorphism.*

Proof. Assume first that $p \in \beta S$, $q \in \delta S$, and $\widetilde{f}(q) \in \delta T$ and suppose that $\widetilde{f}(p * q) \neq \widetilde{f}(p) * \widetilde{f}(q)$. Pick disjoint open neighborhoods U and V of $\widetilde{f}(p * q)$ and $\widetilde{f}(p) * \widetilde{f}(q)$ respectively. Pick $A \in p$ such that $\widetilde{f} \circ \rho_q\big[\,\overline{A}\,\big] \subseteq U$ and $\rho_{\widetilde{f}(q)} \circ \widetilde{f}\big[\,\overline{A}\,\big] \subseteq V$ and pick $x \in A$. Then $\widetilde{f}(x * q) \in U$ and $f(x) * \widetilde{f}(q) \in V$. Since $\lambda_{f(x)}\big(\widetilde{f}(q)\big) \in V$, pick $B \in \widetilde{f}(q)$ such that $\lambda_{f(x)}\big[\,\overline{B \cap \varphi_T\big(f(x)\big)}\,\big] \subseteq V$. Pick $C \in q$ such that $\widetilde{f} \circ \lambda_x\big[\,\overline{C \cap \varphi_S(x)}\,\big] \subseteq U$ and $\widetilde{f}[\,\overline{C}\,] \subseteq \overline{B \cap \varphi_T\big(f(x)\big)}$. Pick $y \in C \cap \varphi_S(x)$. Then $f(x * y) \in U$ and $f(x) * f(y) \in V$, a contradiction.

Now assume that $f[S]$ is an adequate partial subsemigroup of T. Let $p \in \delta S$ and let $x \in T$. Suppose that $\varphi_T(x) \notin \widetilde{f}(p)$. Pick $A \in p$ such that $\widetilde{f}\big[\,\overline{A}\,\big] \cap \overline{\varphi_T(x)} = \emptyset$ and pick $F \in \mathcal{P}_f(f[S])$ such that $\bigcap_{y \in F} \varphi_{f[S]}(y) \subseteq \varphi_T(x)$. Let $G \in \mathcal{P}_f(S)$ be such that $f[G] = F$, and pick $b \in A \cap \bigcap_{a \in G} \varphi_S(a)$. Then $f(b) \in f[A] \cap \bigcap_{y \in F} \varphi_{f[S]}(y) \subseteq f[A] \cap \varphi_T(x)$, a contradiction. The fact that $\widetilde{f}_{|\delta S}$ is a homomorphism now follows from the first assertion. $\square$

The notions of "right ideal", "left ideal", and "ideal" extend naturally to partial semigroups.

1.17 Definition. *Let S be a partial semigroup.*

(a) A subset I of S is a left ideal of S if and only if $x * y \in I$ whenever $x \in S$ and $y \in I \cap \varphi(x)$.

(b) A subset I of S is a right ideal of S if and only if $x * y \in I$ whenever $x \in I$ and $y \in \varphi(x)$.

(c) A subset I of S is an ideal of S if and only if I is both a left ideal and a right ideal of S.

(d) A subset L of S is a minimal left ideal of S if and only if L is a left ideal of S and whenever J is a left ideal of S and $J \subseteq L$ one has $J = L$.

(e) A subset R of S is a minimal right ideal of S if and only if R is a right ideal of S and whenever J is a right ideal of S and $J \subseteq R$ one has $J = R$.

The following are some of the properties of compact right topological semigroups that will be useful for our discussion. These results are taken directly from [9] where they are stated for an arbitrary semigroup. Recall that by Lemma 1.9, δS is a semigroup.

1.18 Lemma. *Let S be an adequate partial semigroup. Let I be an ideal of δS and let L be a minimal left ideal of δS. Then $L \subseteq I$.*

Proof. Since S is adequate $\delta S \neq \emptyset$. Let $x \in L$. Then since I is a right ideal also, $I * x \subseteq I * \delta S \subseteq I$. But $I * x \subseteq \delta S * x \subseteq \delta S * L = L$. So by the minimality of L we have $I * x = L \subseteq I$. □

In Chapter 4 we discuss the notions IP and central. Both of these notions are characterized algebraically in terms of idempotents in δS. Compact right topological semigroups contain idempotents therefore δS does. Hence we have the following theorem.

1.19 Theorem. *Let S be an adequate partial semigroup. Then* *(a)* $E(\delta S) = \{e \in \delta S : \ e * e = e\} \neq \emptyset$.

(b) Every left ideal of δS *contains a minimal left ideal.*

(c) Minimal left ideals of δS *are closed.*

(d) Each minimal left ideal of δS *has an idempotent.*

Proof. (a) Lemma 1.9 and [9, Theorem 2.5].

(b)-(d) [9, Corollary 2.6]. □

1.20 Definition. Let $p = p{*}p \in \delta S$ and let $A \in p$. Then $A^\star = \{x \in A : \ x^{-1}A \in p\}$.

Given an idempotent $p \in \delta S$ and $A \in p$, it is immediate that $A^\star \in p$. The following fact is used in the alternate proof of Theorem 4.6.

1.21 Lemma. *Let* S *be an adequate partial semigroup and let* $A \subseteq S$*. Further let* $p = p * p \in \delta S$*, let* $A \in p$*, and let* $x \in A^\star$*. Then* $x^{-1}(A^\star) \in p$*.*

Proof. Since $x \in A^\star$, $x^{-1}A \in p$ and so $(x^{-1}A)^\star \in p$. Thus it suffices to show that $(x^{-1}A)^\star \subseteq x^{-1}(A^\star)$. (In fact equality holds.) Let $y \in (x^{-1}A)^\star$. Then $y \in x^{-1}A$ and $y^{-1}(x^{-1}A) \in p$. Then $y \in \varphi(x)$ and $x * y \in A$. By Lemma 1.5, we have that $(x * y)^{-1}A = y^{-1}(x^{-1}A) \in p$. So $y \in \varphi(x)$ and $x * y \in A^\star$ as required. □

As is the case for any compact right topological semigroup, δS has a smallest ideal.

1.22 Theorem. *Let* S *be an adequate partial semigroup. Then* δS *has a smallest (two sided) ideal* $K(\delta S)$ *which is the union of all minimal left ideals of* δS *and also the union of all minimal right ideals of* δS*.*

Proof. Lemma 1.9 and [9, Theorem 2.8]. □

1.23 Lemma. *Let* T *be a partial semigroup, let* S *be an adequate partial subsemigroup of* T *and assume that* S *is an ideal of* T*. Then* δS *is an ideal of* δT*. In particular,* $K(\delta S) = K(\delta T)$*.*

Proof. By Lemma 1.10, $\delta S \subseteq \delta T$. Let $p \in \delta S$ and $q \in \delta T$. To see that $q * p \in \delta S$, let $x \in S$. We need to show that $\varphi_S(x) \in q * p$. Since $q \in \delta T$, $\varphi_T(x) \in q$. We claim that $\varphi_T(x) \subseteq \{y \in T : y^{-1}\varphi_S(x) \in p\}$. (Here $y^{-1}\varphi_S(x)$ is interpreted in T, so $y^{-1}\varphi_S(x) = \{z \in \varphi_T(y) : y * z \in \varphi_S(x)\}$.) So let $y \in \varphi_T(x)$ and pick $H \in \mathcal{P}_f(S)$ such that $\bigcap_{z\in H} \varphi_S(z) \subseteq \varphi_T(x * y)$. We claim that $\bigcap_{z\in H} \varphi_S(z) \subseteq y^{-1}\varphi_S(x)$, and thus that $y^{-1}\varphi_S(x) \in p$ as required. So let $w \in \bigcap_{z\in H} \varphi_S(z)$. Then $w \in \varphi_T(x * y)$. So $(x * y) * w$ is defined in T and so $x * (y * w)$ is defined in T. In particular, $y * w \in \varphi_T(x)$. Since $w \in S$, and S is an ideal of T, $y * w \in \varphi_T(x) \cap S = \varphi_S(x)$. Also, $w \in \varphi_T(y)$. Thus, $w \in y^{-1}\varphi_S(x)$.

To see that $p * q \in \delta S$, let $x \in S$. We need to show that $\varphi_S(x) \in p * q$. We claim that $\varphi_S(x) \subseteq \{y \in T : y^{-1}\varphi_S(x) \in q\}$. (Again $y^{-1}\varphi_S(x)$ is interpreted in T.) Let $y \in \varphi_S(x)$. We claim that $\varphi_T(x * y) \subseteq y^{-1}\varphi_S(x)$, so that $y^{-1}\varphi_S(x) \in q$. Let $z \in \varphi_T(x * y)$. Then $(x * y) * z$ is defined in T and so $x * (y * z)$ is defined in T. Therefore $z \in \varphi_T(y)$. And since S is an ideal of T, $y * z \in S$, so $y * z \in \varphi_S(x)$.

Finally, since δS is an ideal of δT, we have that $K(\delta T) \subseteq \delta S$ and in particular $K(\delta T) \cap \delta S \neq \emptyset$. Consequently by [9, Theorem 1.65], $K(\delta S) = K(\delta T) \cap \delta S = K(\delta T)$.

□

CHAPTER II

THICK and SYNDETIC SETS

The first notions we consider are the notions of thick and syndetic. The section begins with a reminder of what each notion means in the context of a semigroup S along with their algebraic characterizations as stated in [3]. We then extend these notions to an adequate partial semigroup, thereby obtaining analogous statements for the combinatorial definition and the algebraic characterization of each notion for an adequate partial semigroup. The main results of this chapter show that the resulting extensions correspond to nonequivalent notions in an adequate partial semigroup.

Thick sets are intimately related to syndetic sets in a semigroup. We find that in an adequate partial semigroup this intimacy is maintained in a significant way among both the algebraic and the combinatorial versions. We introduce some of these interrelationships towards the middle of the chapter, as a bridge between the notions.

Thick and syndetic sets are algebraically characterized in βS in terms of the left ideals of βS. Given $p \in \beta S$, $\beta S \cdot p$ is a left ideal of βS. Further, given any left ideal L of βS and any $p \in L$, $\beta S \cdot p \subseteq L$. A left ideal of the form $\beta S \cdot p$ is called a *semiprincipal* left ideal. (It is only semiprincipal because p need not be a member of $\beta S \cdot p$.) In the case of an adequate partial semigroup, we are naturally interested in the left ideals of δS of the form $\delta S * p$ for $p \in \delta S$.

For a semigroup S, the semiprincipal left ideals of βS are categorized in terms of the set $C(p)$ [9]. We construct an analog of the set $C(p)$ for an adequate partial semigroup and use it to categorize the members of $\delta S * p$. We first remind the reader of the definition of $C(p)$ and the characterization of the semiprincipal left ideals of βS in terms of $C(p)$.

2.1 Definition. Let S be a semigroup and let $p \in \beta S$. Then

$$C(p) = \{A \subseteq S : \text{for all } s \in S,\ s^{-1}A \in p\}.$$

2.2 Theorem. *Let S be a semigroup and let $p \in \beta S$. Then*

$$\beta S \cdot p = \{q \in \beta S : C(p) \subseteq q\}.$$

Proof. [9, Theorem 6.18]. □

In the partial semigroup case we define the set $D(p)$ as the analog of $C(p)$. Like $C(p)$, $D(p)$ is a filter on the partial semigroup, and characterizes the semiprincipal left ideals of δS.

2.3 Definition. Let $(S, *)$ be an adequate partial semigroup and let $p \in \delta S$. Then $D(p) = \{A \subseteq S : \text{there exists } F \in \mathcal{P}_f(S) \text{ such that for all } x \in \sigma(F),\ x^{-1}A \in p\}$.

Notice that if S is a semigroup $D(p)$ is equal to $C(p)$.

2.4 Theorem. *Let $(S, *)$ be an adequate partial semigroup and let $p \in \delta S$. Then $D(p)$ is a filter on S.*

Proof. $D(p) \neq \emptyset$ since $S \in D(p)$. It is clear that $\emptyset \notin D(p)$, otherwise there exists $H \in \mathcal{P}_f(S)$ such that for all $x \in \sigma(H)$, $x^{-1}\emptyset \in p$. This is impossible.

Assume that A and B are in $D(p)$. To see that $A \cap B \in D(p)$, pick $F, G \in \mathcal{P}_f(S)$ such that for all $x \in \sigma(F)$, $x^{-1}A \in p$ and for all $y \in \sigma(G)$, $y^{-1}B \in p$. Note that $F \cup G \in \mathcal{P}_f(S)$. Let $z \in \sigma(F \cup G)$. Then $z \in \sigma(F)$ and $z \in \sigma(G)$. So $z^{-1}A \in p$ and $z^{-1}B \in p$. Therefore $z^{-1}A \cap z^{-1}B = z^{-1}(A \cap B) \in p$. So $A \cap B \in D(p)$.

Let $A \in D(p)$ and let $B \subseteq S$ such that $A \subseteq B$. Pick $F \in \mathcal{P}_f(S)$ such that for all $x \in \sigma(F)$, $x^{-1}A \in p$. So given $x \in \sigma(F)$, $\{s \in \varphi(x) : x * s \in A\} \in p$ and $\{s \in \varphi(x) : x * s \in A\} \subseteq \{s \in \varphi(x) : x * s \in B\}$, so $\{s \in \varphi(x) : x * s \in B\} \in p$. Therefore for all $x \in \sigma(F)$, $x^{-1}B \in p$. Thus $B \in D(p)$.

So $D(p)$ is a filter on S. □

2.5 Theorem. *Let $(S,*)$ be an adequate partial semigroup and let $p \in \delta S$. Then $\delta S * p = \{q \in \delta S : D(p) \subseteq q\}$.*

Proof. Let $r \in \delta S$. We show that $D(p) \subseteq r * p$. Let $A \in D(p)$ and pick $F \in \mathcal{P}_f(S)$ such that for all $x \in \sigma(F)$, $x^{-1}A \in p$. So $\sigma(F) = \{x \in \sigma(F) : x^{-1}A \in p\}$. Also $r \in \delta S$ so $\sigma(F) \in r$. Therefore $\{x \in \sigma(F) : x^{-1}A \in p\} \in r$. Thus $A \in r * p$.

Now let $q \in \delta S$ such that $D(p) \subseteq q$. For each $A \in q$, let $B(A) = \{x \in S : x^{-1}A \in p\}$. We claim that $\{B(A) : A \in q\} \cup \{\sigma(F) : F \in \mathcal{P}_f(S)\}$ has the finite intersection property. To see this, first observe that $\{B(A) : A \in q\}$ and $\{\sigma(F) : F \in \mathcal{P}_f(S)\}$ are both closed under finite intersections. So it suffices to show that given $A \in q$ and $F \in \mathcal{P}_f(S)$, $B(A) \cap \sigma(F) \neq \emptyset$. Suppose instead that there exist $A \in p$ and $F \in \mathcal{P}_f(S)$ such that $B(A) \cap \sigma(F) = \emptyset$. Then $\{x \in \sigma(F) : x^{-1}A \in p\} = \emptyset$. That is, for each $x \in \sigma(F)$, $x^{-1}(S \backslash A) \in p$. So $S \backslash A \in D(p) \subseteq q$. This is a contradiction. So $\{B(A) : A \in q\} \cup \{\sigma(F) : F \in \mathcal{P}_f(S)\}$ has the finite intersection property. Therefore pick $r \in \beta S$ such that $\{B(A) : A \in q\} \cup \{\sigma(F) : F \in \mathcal{P}_f(S)\} \subseteq r$. Since $\{\sigma(F) : F \in \mathcal{P}_f(S)\} \subseteq r$, $r \in \delta S$. Since $\{x \in S : x^{-1}A \in p\} \in r$ for each $A \in q$, $q = r * p$. Therefore $q \in \delta S * p$. □

2.6 Theorem. *Let $(S,*)$ be an adequate partial semigroup and let $p \in \delta S$. Then $D(p) = \bigcap(\delta S * p)$.*

Proof. Let $A \in D(p)$, and pick $F \in \mathcal{P}_f(S)$ such that for all $x \in \sigma(F)$, $x^{-1}A \in p$. Suppose that $A \notin \bigcap(\delta S * p)$. Pick $q \in \delta S$ such that $A \notin q * p$. Then $\{x \in S : x^{-1}A \in p\} \notin q$. Equivalently $\{x \in S : x^{-1}A \notin p\} \in q$. Also $\sigma(F) \in q$ so $\{x \in \sigma(F) : x^{-1}A \notin p\} \in q$. But $\{x \in \sigma(F) : x^{-1}A \notin p\} = \emptyset$. This is a contradiction.

Now let $A \in \bigcap(\delta S * p)$, and suppose that $A \notin D(p)$. Then $\{S \backslash A\} \cup D(p)$ has the finite intersection property because $D(p)$ is a filter. So pick $q \in \beta S$ such that

$S\backslash A \in q$ and $D(p) \subseteq q$. Then by Theorem 2.5, $q \in (\delta S * p) \cap \overline{(S\backslash A)}$. This is a contradiction since by assumption $A \in q$. □

2.7 Theorem. *Let $(S, *)$ be an adequate partial semigroup, let $A \subseteq S$, and let $p \in \delta S$. Then $\delta S * p \subseteq \overline{A}$ if and only if $A \in D(p)$.*

Proof. We use Theorem 2.6.

$$\begin{aligned} A \in D(p) &\Leftrightarrow A \in \bigcap(\delta S * p) \\ &\Leftrightarrow (\forall q \in \delta S)(A \in q * p) \\ &\Leftrightarrow (\forall q \in \delta S)(q * p \in \overline{A}\,) \\ &\Leftrightarrow \delta S * p \subseteq \overline{A} \end{aligned}$$

□

The last result takes us to our discussion of the notion of thick sets in an adequate partial semigroup. By Theorem 2.7, we shall see that a set is thick if and only if it is a member of $D(p)$. Therefore we will return to this result as a characterization of thick sets in δS.

The notion of a thick set originates in Topological Dynamics in the context of the semigroup $(\mathbb{N}, +)$. There [6], a set is said to be thick if it contains arbitrarily long intervals $(a_n, a_n + n)$, where $n \to \infty$. More recently, thick sets (referred to as right thick) were presented in the context of compact right topological semigroups [3]. It was shown that thick sets are easily characterized in terms of βS, the Stone-Čech compactification of S.

2.8 Definition. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. The set A is *thick* if and only if for every $F \in \mathcal{P}_f(S)$ there is some $x \in S$ such that $Fx \subseteq A$.

2.9 Theorem. *Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. The following statements are equivalent.*

(a) The set A is thick.

(b) There exists $p \in \beta S$ such that $\beta S \cdot p \subseteq \overline{A}$.

(c) There is a left ideal L of βS with $L \subseteq \overline{A}$.

Proof. $(a) \Rightarrow (b)$. Since A is thick, $\{t^{-1}A : t \in S\}$ has the finite intersection property. Pick $p \in \beta S$ such that $\{t^{-1}A : t \in S\} \subseteq p$. Then $S \cdot p \subseteq \overline{A}$ and thus $\beta S \cdot p \subseteq \overline{A}$.

That (b) implies (c) is trivial.

$(c) \Rightarrow (a)$. Pick a left ideal L of βS such that $L \subseteq \overline{A}$ and pick $p \in L$. Then for each $t \in S$, $t \cdot p \in \overline{A}$ and so $t^{-1}A \in p$. Given $F \in \mathcal{P}_f(S)$, pick $x \in \bigcap_{t \in F} t^{-1}A$. □

The notion of a syndetic set also originates in Topological Dynamics in the context of the semigroup $(\mathbb{N}, +)$. In this context a set A is syndetic if and only if it has bounded gaps.

In [3], syndetic sets, (referred to as right syndetic), in an abitrary semigroup S, were similarly characterized in terms of βS.

2.10 Definition. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. The set A is *syndetic* if and only if there exists $H \in \mathcal{P}_f(S)$ such that $S \subseteq \bigcup_{t \in H} t^{-1}A$.

2.11 Theorem. *Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. The following statements are equivalent.*

(a) The set A is syndetic.

(b) For each $p \in \beta S$, $\overline{A} \cap (\beta S \cdot p) \neq \emptyset$.

(c) For each left ideal L of βS, $\overline{A} \cap L \neq \emptyset$.

Proof. $(a) \Rightarrow (b)$. Let $p \in \beta S$ and pick $H \in \mathcal{P}_f(S)$ such that $S \subseteq \bigcup_{t \in H} t^{-1}A$. Pick $t \in H$ such that $t^{-1}A \in p$. Then $t \cdot p \in (\beta S \cdot p) \cap \overline{A}$.

That (b) implies (c) is trivial.

$(c) \Rightarrow (a)$. Suppose that for each $H \in \mathcal{P}_f(S)$, $S \backslash \bigcup_{t \in H} t^{-1}A \neq \emptyset$. Then $\{S \backslash t^{-1}A : t \in S\}$ has the finite intersection property so pick $p \in \beta S$ such that $\{S \backslash t^{-1}A : t \in S\} \subseteq p$. Then $\overline{A} \cap (\beta S \cdot p) \neq \emptyset$. So pick $q \in \beta S$ such that $A \in q \cdot p$. Then $\{t \in S : t^{-1}A \in p\} \in q$ so for some $t \in S$, $t^{-1}A \in p$, a contradiction. □

The algebraic characterization of thick has a completely straight forward extension to partial semigroups.

2.12 Definition. Let $(S,*)$ be an adequate partial semigroup and let $A \subseteq S$. The set A is thick if and only if there exists $p \in \delta S$ such that $\delta S * p \subseteq \overline{A}$.

In an adequate partial semigroup $(S,*)$, given a finite subset F of S, $F * x$ need not be defined for every member of F. Therefore, the combinatorial definition of thick stated above, cannot be transferred verbatim to the partial semigroup case. The modification needed for the extension to partial semigroups is minor, as indicated below. (The notation $\check{c}$-thick is intended to represent "combinatorially thick". We will adhere to this notation for all combinatorially defined notions.)

2.13 Definition. Let $(S,*)$ be an adequate partial semigroup and let $A \subseteq S$. A is $\check{c}$-thick if and only if for each $F \in \mathcal{P}_f(S)$, there exists $x \in \sigma(F)$ such that $F*x \subseteq A$.

First we show that these two notions are not equivalent. The example given uses an adequate partial semigroup introduced in Chapter 1. We will use this partial semigroup again in this and subsequent chapters.

2.14 Theorem. *There exist an adequate partial semigroup $(T,*)$ and a thick subset A of T which is not $\check{c}$-thick.*

Proof. Let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in a semigroup $(S,\cdot)$ which satisfies uniqueness of finite products (meaning $\prod_{n\in F} x_n = \prod_{n\in G} x_n$, with products in increasing order of indices, only when $F = G$). (For example S could be the free semigroup on the generators $\langle x_n\rangle_{n=1}^{\infty}$. Alternatively, one could take $(S,\cdot) = (\mathbb{N},+)$ and let $x_n = 2^n$.)

Let $T = FP(\langle x_n\rangle_{n=1}^{\infty}) = \{\prod_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$ and define

$$\left(\textstyle\prod_{n\in F} x_n\right) * \left(\textstyle\prod_{n\in G} x_n\right) = \begin{cases} \prod_{n\in F\cup G} x_n & \text{if } \max F < \min G \\ \text{undefined} & \text{if } \max F \geq \min G\,. \end{cases}$$

Let $A = \{\prod_{n\in F} x_n : \min F > 1\} = FP(\langle x_n\rangle_{n=2}^{\infty})$. Then $A = \varphi(x_1)$ and thus $\delta S \subseteq \overline{A}$. In particular, for any $p \in \delta S$, $\delta S * p \subseteq \delta S \subseteq \overline{A}$ and so A is thick.

To see that A is not $\check{c}$-thick, let $F = \{x_1\}$. Then for any $y \in \sigma(F) = A$, $F * y \subseteq (S \backslash A)$. $\square$

We shall soon show that the notion of thick in an adequate partial semigroup is weaker than the notion of $\check{c}$-thick. To produce a simple proof of this fact, we first construct the algebraic characterization of a $\check{c}$-thick set, in terms of βS.

2.15 Theorem. *Let $(S, *)$ be an adequate partial semigroup and let $A \subseteq S$. Then A is $\check{c}$-thick if and only if there exists $p \in \delta S$ such that $\beta S * p \subseteq \overline{A}$.*

Proof. Assume first that A is $\check{c}$-thick. We claim that $\{\sigma(F) : F \in \mathcal{P}_f(S)\} \cup \{x^{-1}A : x \in S\}$ has the finite intersection property. Since $\{\sigma(F) : F \in \mathcal{P}_f(S)\}$ is closed under finite intersections, it suffices to let $F, H \in \mathcal{P}_f(S)$ and show that $\sigma(F) \cap \bigcap_{t \in H} t^{-1}A \neq \emptyset$. To see this, let $G = F \cup H$. Then $G \in \mathcal{P}_f(S)$ so we can pick $x \in \sigma(G)$ such that $G * x \subseteq A$. Then $x \in \sigma(F)$, $x \in \sigma(H)$, and $H * x \subseteq A$. That is, $x \in \sigma(F)$ and for all $t \in H$, we have that $x \in \varphi(t)$ and $t * x \in A$. So $x \in \sigma(F) \cap \bigcap_{t \in H} t^{-1}A$. Thus that claim holds. Therefore we can pick $p \in \beta S$ such that $\{\sigma(F) : F \in \mathcal{P}_f(S)\} \cup \{x^{-1}A : x \in S\} \subseteq p$. Since $\{\sigma(F) : F \in \mathcal{P}_f(S)\} \subseteq p$ we have $p \in \delta S$. To see that $\beta S * p \subseteq \overline{A}$, let $q \in \beta S$. Since $S = \{t \in S : t^{-1}A \in p\}$, $\{t \in S : t^{-1}A \in p\} \in q$ and so $A \in q * p$.

Now pick $p \in \delta S$ such that $\beta S * p \subseteq \overline{A}$. We want to show that given $F \in \mathcal{P}_f(S)$, there exists $t \in \sigma(F)$ such that $F * t \subseteq A$. To see this, let $F \in \mathcal{P}_f(S)$. Since $\beta S * p \subseteq \overline{A}$, for any $x \in S$ we have that $A \in x * p$. In particular, for all $x \in F$ we have that $x^{-1}A \in p$. Since $p \in \delta S$, $\sigma(F) \in p$. Therefore $\sigma(F) \cap \bigcap_{x \in F} x^{-1}A \neq \emptyset$. So pick $t \in \sigma(F) \cap \bigcap_{x \in F} x^{-1}A$. Then $F * t \subseteq A$. So A is $\check{c}$-thick. $\square$

We also provide an equivalent combinatorial statement of the definition of a thick set in an adequate partial semigroup.

2.16 Theorem. *Let $(S, *)$ be an adequate partial semigroup and let $A \subseteq S$. The set A is thick if and only if there exists $F \in \mathcal{P}_f(S)$ such that for all $H \in \mathcal{P}_f(S)$ and* *for all* $L \in \mathcal{P}_f\big(\sigma(F)\big)$ *there exists* $x \in \sigma(H \cup L)$ *such that* $L * x \subseteq A$.

Proof. Necessity. Assume A is thick and pick $p \in \delta S$ such that $\delta S * p \subseteq \overline{A}$. Then by Theorem 2.7, $A \in D(p)$. So pick $F \in \mathcal{P}_f(S)$ such that for all $x \in \sigma(F), x^{-1}A \in p$. Let $H \in \mathcal{P}_f(S)$ and $L \in \mathcal{P}_f\big(\sigma(F)\big)$ be given. Then $\bigcap_{x\in L} x^{-1}A \in p$, and $\sigma(H) \in p$, so $\bigcap_{x\in L} x^{-1}A \cap \sigma(H) \in p$. Thus $\bigcap_{x\in L} x^{-1}A \cap \sigma(H) \neq \emptyset$. Pick $y \in \bigcap_{x\in L} x^{-1}A \cap \sigma(H)$. Since $y \in \bigcap_{x\in L} x^{-1}A$, $y \in \sigma(H \cup L)$. Also $L * y \subseteq A$.

Sufficiency. Pick $F \in \mathcal{P}_f(S)$ such that for all $H \in \mathcal{P}_f(S)$ and for all $L \in \mathcal{P}_f\big(\sigma(F)\big)$ there exists $x \in \sigma(H \cup L)$ such that $L * x \subseteq A$. We claim that $\{\sigma(H) : H \in \mathcal{P}_f(S)\} \cup \{t^{-1}A : t \in \sigma(F)\}$ has the finite intersection property. To see that the claim is true, let $H \in \mathcal{P}_f(S)$ and let $L \in \mathcal{P}_f\big(\sigma(F)\big)$. By assumption we have an $x \in \sigma(H \cup L)$ such that $L * x \subseteq A$. So $x \in \bigcap_{t\in L} t^{-1}A$, and thus $\bigcap_{t\in L} t^{-1}A \cap \sigma(H) \neq \emptyset$. So $\{\sigma(H) : H \in \mathcal{P}_f(S)\} \cup \{t^{-1}A : t \in \sigma(F)\}$ has the finite intersection property. Therefore we can pick $p \in \beta S$ such that $\{\sigma(H) : H \in \mathcal{P}_f(S)\} \cup \{t^{-1}A : t \in \sigma(F)\} \subseteq p$. Since $\{\sigma(H) : H \in \mathcal{P}_f(S)\} \subseteq p$ we have that $p \in \delta S$. We now claim that $\delta S * p \subseteq \overline{A}$. To see this, let $q \in \delta S$. Then $\sigma(F) \in q$. Since $\{t^{-1}A : t \in \sigma(F)\} \subseteq p$, we have that $\sigma(F) \subseteq \{t \in S : t^{-1}A \in p\}$. So $\{t \in S : t^{-1}A \in p\} \in q$. Therefore $A \in q * p$. □

As an immediate consequence of Theorem 2.15 we have the following implication.

2.17 Theorem. *Let* $(S, *)$ *be an adequate partial semigroup, and let* $A \subseteq S$. *If* A *is a č-thick set, then* A *is thick set.*

Proof. The set A is č-thick so by Theorem 2.15, we can pick $p \in \delta S$ such that $\beta S * p \subseteq \overline{A}$. Since $\delta S * p \subseteq \beta S * p$, we have that $\delta S * p \subseteq \overline{A}$. So A is thick. □

We observe that č-thickness is equivalent to a superficially stronger statement.

2.18 Lemma. *Let* $(S, *)$ *be an adequate partial semigroup and let* $A \subseteq S$. *If* A *is č-thick, then for every* $F \in \mathcal{P}_f(S)$ *there exists* $x \in A \cap \sigma(F)$ *such that* $F * x \subseteq A$.

Proof. Let $H \in \mathcal{P}_f(S)$ and pick $t \in \sigma(H)$. Then $F = (H * t) \cup \{t\} \in \mathcal{P}_f(S)$. Since A is č-thick we can pick $x \in \sigma(F)$ such that $F * x \subseteq A$. So $t * x \in \sigma(H)$, $t * x \in A$, and $H * (t * x) \subseteq A$. □

We turn our attention now to the notion of a syndetic set. The combinatorial definition and the algebraic characterization of syndetic in a semigroup can both be extended to partial semigroups in a natural way.

2.19 Definition. Let $(S, *)$ be an adequate partial semigroup and let $A \subseteq S$. The set A is *syndetic* if and only if for every $p \in \delta S$, $\overline{A} \cap (\delta S * p) \neq \emptyset$.

Since $*$ is defined for only a subset of S, we are unlikely to find a finite subset H of S such that $S \subseteq \bigcup_{t \in H} \varphi(t)$. Thus we cannot transfer, verbatim, the definition for syndetic to partial semigroups. However, a minor adjustment is sufficient.

2.20 Definition. Let $(S, *)$ be an adequate partial semigroup and let $A \subseteq S$. Then A is *č-syndetic* if and only if there exists $H \in \mathcal{P}_f(S)$ such that $\sigma(H) \subseteq \bigcup_{t \in H} t^{-1}A$.

Note that the combinatorial definition of syndetic in a partial semigroup, (č-syndetic), guarantees that S itself is syndetic. The notions "syndetic" and "č-syndetic" are not equivalent, though we shall see that every syndetic subset of an adequate partial semigroup is also č-syndetic. As an example of a set which is č-syndetic but not syndetic, we have the following.

2.21 Theorem. *There exist an adequate partial semigroup $(T, *)$ and a č-syndetic subset A of T which is not syndetic. In fact, for any $p \in \delta S$, $\overline{A} \cap (\delta S * p) = \emptyset$.*

Proof. Let T be the partial semigroup introduced in the proof of Theorem 2.14. Let $A = \{\prod_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N}) \text{ and } 1 \in F\}$. To see that A is č-syndetic, let $H = \{x_1\}$, so that $\sigma(H) = \varphi(x_1) = \{\prod_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N}) \text{ and } \min F > 1\}$. Then $\sigma(H) \subseteq {x_1}^{-1}A$.

Now let $p,\ q \in \delta S$ and suppose that $A \in q * p$. Then $\{x \in S : x^{-1}A \in p\} \in q$ and $\varphi(x_1) \in q$ so pick $y \in \varphi(x_1)$ such that $y^{-1}A \in p$. Pick $z \in y^{-1}A$. Then $z \in \varphi(y)$ and $y * z \in A$. But $y \in \varphi(x_1)$, so $y = \prod_{n \in F} x_n$ where $\min F > 1$. Thus $y * z \notin A$. This is a contradiction. So A is not syndetic. □

As mentioned at the beginning of this chapter, in a semigroup thick sets and syndetic sets are intimately related. A subset A of a semigroup S is thick if and only if its complement is not syndetic. Motivated by this result, which is the basis of the proof of Theorem 2.21, we are able to produce analogous results for the partial semigroup case. These results become helpful in proving many other results and so we introduce them here. As always, we begin by stating the known results for the semigroup case, taken from [3].

2.22 Theorem. *Let $(S\cdot)$ be a semigroup and let $A \subseteq S$. Then the following are equivalent.*

(a) The set A is thick.

(b) The set A intersects every syndetic set nontrivially.

(c) The set $S \backslash A$ is not syndetic.

Proof. $(a) \Rightarrow (b)$. Let B be a syndetic set. Pick $H \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in H} t^{-1}B$. Pick $x \in S$ such that $H \cdot x \subseteq A$ and pick $t \in H$ such that $t \cdot x \in B$. Then $t \cdot x \in A \cap B$.

$(b) \Rightarrow (c)$. We have that $A \cap (S \backslash A) = \emptyset$.

$(c) \Rightarrow (a)$. To see that A is thick, let $F \in \mathcal{P}_f(S)$. Since $S \backslash A$ is not syndetic, pick $x \in S \backslash \bigcup_{t \in F} t^{-1}(S \backslash A)$. Then $F \cdot x \subseteq A$. □

Though the notions of thick (respectively syndetic) and $\check{c}$-thick (respectively $\check{c}$-syndetic) are not equivalent we find that in an adequate partial semigroup, the relationship between thick and syndetic sets is exactly the relationship between $\check{c}$-thick and $\check{c}$-syndetic sets.

2.23 Theorem. *Let $(S,*)$ be an adequate partial semigroup and let $A \subseteq S$. Then the following are equivalent.*

(a) The set A is thick.

(b) The set A intersects every syndetic set nontrivially.

(c) The set $S \backslash A$ is not syndetic.

Proof. $(a) \Rightarrow (b)$. Let B be a syndetic subset of S. Since A is thick, pick $p \in \delta S$ such that $\delta S * p \subseteq \overline{A}$. Since B is syndetic $(\delta S * p) \cap \overline{B} \neq \emptyset$. Therefore $\overline{A} \cap \overline{B} = \overline{A \cap B} \neq \emptyset$.

$(b) \Rightarrow (c)$. We have that $A \cap (S \backslash A) = \emptyset$.

$(c) \Rightarrow (a)$. Pick $p \in \delta S$ such that $\overline{(S \backslash A)} \cap (\delta S * p) = \emptyset$. Then $\delta S * p \subseteq \overline{A}$. $\square$

2.24 Theorem. *Let $(S,*)$ be an adequate partial semigroup and let $A \subseteq S$. Then the following are equivalent.*

(a) The set A is č-thick.

(b) The set A intersects every č-syndetic set nontrivially.

(c) The set $S \backslash A$ is not č-syndetic.

Proof. $(a) \Rightarrow (b)$. Let B be a č-syndetic subset of S. The set B is č-syndetic so pick $H \in \mathcal{P}_f(S)$ such that $\sigma(H) \subseteq \bigcup_{t \in H} t^{-1}B$. Also pick $x \in \sigma(H)$ such that $H * x \subseteq A$. Pick $t \in H$ such that $x \in t^{-1}B$. Then $t * x \in (A \cap B)$.

(b) $\Rightarrow$ (c) We have that $A \cap (S \backslash A) = \emptyset$.

(c) $\Rightarrow$ (a) Let $H \in \mathcal{P}_f(S)$, and pick $x \in \sigma(H)$ such that $x \notin \bigcup_{t \in H} t^{-1}(S \backslash A)$. Then for any $t \in H$, $t * x \notin S \backslash A$. So for all $t \in H$, $t * x \in A$. Therefore $H * x \subseteq A$. So A is č-thick. $\square$

We obtain now an algebraic characterization of "č-syndetic" and a combinatorial characterization of "syndetic", based on the corresponding characterizations of "č-thick" and "thick".

2.25 Theorem. *Let $(S,*)$ be an adequate partial semigroup and let $A \subseteq S$. Then A is č-syndetic if and only if for all $p \in \delta S$, $\overline{A} \cap (\beta S * p) \neq \emptyset$.*

Proof. Necessity. The set A is č-syndetic so by Theorem 2.24 $S \backslash A$ is not č-thick. Then by Theorem 2.15, for each $p \in \delta S$, we have that $\beta S * p \nsubseteq \overline{(S \backslash A)}$. Thus, for all $p \in \delta S$, $\overline{A} \cap (\beta S * p) \neq \emptyset$.

Sufficiency. Suppose that A is not č-syndetic. By Theorem 2.24 $S \backslash A$ is č-thick so pick by Theorem 2.15 some $p \in \delta S$ such that $\beta S * p \subseteq \overline{S \backslash A}$. Then $\overline{A} \cap (\beta S * p) = \emptyset$, a contradiction. □

2.26 Theorem. *Let $(S,*)$ be an adequate partial semigroup and let $A \subseteq S$. Then A is syndetic if and only if for each $F \in \mathcal{P}_f(S)$ there exist $L \in \mathcal{P}_f\big(\sigma(F)\big)$ and $H \in \mathcal{P}_f(S)$ such that $\sigma(H \cup L) \subseteq \bigcup_{t \in L} t^{-1}A$.*

Proof. Necessity. The set A is syndetic so by Theorem 2.23 $S \backslash A$ is not thick. Let $F \in \mathcal{P}_f(S)$. By Theorem 2.16 we can pick $H \in \mathcal{P}_f(S)$ and $L \in \mathcal{P}_f\big(\sigma(F)\big)$ such that for each $x \in \sigma(H \cup L)$, $L * x \nsubseteq S \backslash A$. To see that $\sigma(H \cup L) \subseteq \bigcup_{t \in L} t^{-1}A$, let $x \in \sigma(H \cup L)$. Pick $t \in L$ such that $t * x \notin S \backslash A$. Then $x \in t^{-1}A$.

Sufficiency. Suppose A is not syndetic. Then by Theorem 2.23 $S \backslash A$ is thick. So by Theorem 2.16 we can pick $F \in \mathcal{P}_f(S)$ such that for all $H \in \mathcal{P}_f(S)$ and for all $L \in \mathcal{P}_f\big(\sigma(F)\big)$ there exists $x \in \sigma(H \cup L)$ such that $L * x \subseteq S \backslash A$. By assumption, choose $L \in \mathcal{P}_f\big(\sigma(F)\big)$ and $H \in \mathcal{P}_f(S)$ such that $\sigma(H \cup L) \subseteq \bigcup_{t \in L} t^{-1}A$. Pick $x \in \sigma(H \cup L)$ such that $L * x \subseteq S \backslash A$. Picking $t \in L$ such that $x \in t^{-1}A$, we have a contradiction. □

As a consequence of Theorem 2.25 we have the following.

2.27 Theorem. *Let $(S,*)$ be an adequate partial semigroup and let $A \subseteq S$. If A is syndetic then A is č-syndetic.*

Proof. Assume that $A \subseteq S$ is syndetic. Let $p \in \delta S$. Then $\overline{A} \cap (\delta S * p) \neq \emptyset$. And $\delta S \subseteq \beta S$, so $\overline{A} \cap (\beta S * p) \neq \emptyset$. So by Theorem 2.25 A is č-syndetic. □

Though the notions of "syndetic" and "č-syndetic" are not equivalent, they play an identical role in the characterization of members of the smallest ideal, $K(\delta S)$. The fact that one is able to characterize the members of $K(\delta S)$ is important because of the extensive structure which this ideal is known to have.

The proof that (c) implies (d) in the following theorem is taken from [8, Theorem 2.15].

2.28 Theorem. *Let $(S,*)$ be an adequate partial semigroup and let $p \in \delta S$. The following statements are equivalent:*

(a) $p \in K(\delta S)$.

(b) For all $A \in p$, $\{x \in S : x^{-1}A \in p\}$ is syndetic.

(c) For all $A \in p$, $\{x \in S : x^{-1}A \in p\}$ is č-syndetic.

(d) For all $q \in \delta S$, $p \in \delta S * q * p$.

Proof. $(a) \Rightarrow (b)$. Let $A \in p$ and let $B = \{x \in S : x^{-1}A \in p\}$. Let L be a minimal left ideal of δS with $p \in L$. We show that for every left ideal L' of δS, $\overline{B} \cap L' \neq \emptyset$. Let L' be a left ideal of δS. Then $L' * p$ is a left ideal of δS and $L' * p \subseteq L$ because L is a left ideal. So $L' * p = L$ (by the minimality of L). Pick $q \in L'$ such that $p = q * p$. Since $A \in p = q * p$, $B = \{x \in S : x^{-1}A \in p\} \in q$ and so $q \in \overline{B}$.

$(b) \Rightarrow (c)$. Theorem 2.27.

$(c) \Rightarrow (d)$. Let $q \in \delta S$. For $A \in p$, let $B(A) = \{x \in S : x^{-1}A \in q * p\}$. We claim that $\{B(A) : A \in p\}$ has the finite intersection property. Since, given A_1 and A_2, $B(A_1 \cap A_2) = B(A_1) \cap B(A_2)$, it suffices to show that each $B(A) \neq \emptyset$. To this end, let $A \in p$, let $C = \{x \in S : x^{-1}A \in p\}$, and pick $H \in \mathcal{P}_f(S)$ such that $\sigma(H) \subseteq \bigcup_{t \in H} t^{-1}C$. For each $y \in \sigma(H)$, pick $t_y \in H$ such that $t_y * y \in C$. Now $\sigma(H) \in q$ and $\sigma(H) = \bigcup_{t \in H}\{y \in \sigma(H) : t_y = t\}$, so pick $t \in H$ such that $\{y \in \sigma(H) : t_y = t\} \in q$. We show that $t \in B(A)$. For this it suffices to show that $\{y \in \sigma(H) : t_y = t\} \subseteq \{y \in S : y^{-1}(t^{-1}A) \in p\}$. So let $y \in \sigma(H)$ such that $t_y = t$.

Then $t * y \in C$ so $(t * y)^{-1}A \in p$. Since $y \in \varphi(t)$, $(t * y)^{-1}A = y^{-1}(t^{-1}A)$.

Since $\{B(A) : A \in p\}$ has the finite intersection property, pick $r \in \beta S$ such that $\{B(A) : A \in p\} \subseteq r$. Then for all $A \in p$, $\{x \in S : x^{-1}A \in q * p\} \in r$ so $p = r * (q * p)$. Since $p \in \delta S$, for each $a \in S$ we have that $\varphi(a) \in p$. Consequently by Lemma 1.8 we have that for each $a \in S$, $\varphi(a) \in r$. That is $r \in \delta S$.

$(d) \Rightarrow (a)$. Pick any $q \in H(\delta S)$. $\square$

CHAPTER III

PIECEWISE SYNDETICITY

There is much written about the smallest ideal of a compact right topological semigroup. We know for instance that for a semigroup S, the ultrafilters in the smallest ideal of βS, $K(\beta S)$, are precisely those whose members are piecewise syndetic sets. Being itself a compact topological semigroup, δS has a smallest ideal, $K(\delta S)$. Naturally we are interested in extending the notion of piecewise syndetic to adequate partial semigroup, and subsequently examining $K(\delta S)$.

As with the other notions we have looked at so far, the notion of *piecewise syndetic* is also borrowed from Topological Dynamics. According to [6], in $\mathbb{N}$, a set A is piecewise syndetic if and only if there exists a fixed bound l and arbitrarily long intervals in which the gaps of A are bounded by l. In the last chapter we looked at thick and syndetic sets. Piecewise syndetic sets proceed naturally from these dual notions. In a semigroup, the intersection of a thick set with a syndetic set is a piecewise syndetic set. This occurrence carries over only partially (pun intended) to adequate partial semigroups.

In [3], piecewise syndetic sets (referred to as right piecewise syndetic) are defined in terms of compact right topological semigroups. There a set A is said to be right piecewise syndetic if and only if there exists $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t\in H} t^{-1}A$ is right thick. This definition is equivalent to the one given below as stated in [9]. The main results of this chapter are motivated by the characterization, given in [3, Theorem 2.9] and [9, Theorem 4.40], of piecewise syndetic sets in terms of $K(\beta S)$. We begin by first extending the notion of piecewise syndetic to an adequate partial semigroup. We show that, as in the last chapter, the extension of the combinatorial and algebraic meanings of piecewise syndetic in a semigroup to an adequate partial semigroup yield two new nonequivalent notions.

Recall the definition and characterization of piecewise syndetic in a semigroup.

3.1 Definition. Let $(S,\cdot)$ be a semigroup. A set $A \subseteq S$ is *piecewise syndetic* if and only if there exists $H \in \mathcal{P}_f(S)$ such that for all $T \in \mathcal{P}_f(S)$, there exists $x \in S$ such that $T \cdot x \subseteq \bigcup_{t \in H} t^{-1}A$.

Alternatively stated, A is piecewise syndetic if and only if there exists $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t \in H} t^{-1}A$ is thick.

3.2 Theorem. *Let $(S,\cdot)$ be a semigroup and let $A \subseteq S$. Then A is piecewise syndetic if and only if $\overline{A} \cap K(\beta S) \neq \emptyset$.*

Proof. Sufficiency. Let $p \in K(\beta S) \cap \overline{A}$ and let $B = \{x \in S : x^{-1}A \in p\}$. Then by Theorem 2.28, B is syndetic and so $S = \bigcup_{t \in H} t^{-1}B$ for some $H \in \mathcal{P}_f(S)$. Now let $T \in \mathcal{P}_f(S)$ be given. For each $a \in T$, $a \in t^{-1}B$ for some $t \in H$. So $a^{-1}(t^{-1}A) = (ta)^{-1}A \in p$ and thus $a^{-1}\big(\bigcup_{t \in H} t^{-1}A\big) \in p$. Pick $x \in \bigcap_{a \in T} a^{-1}\big(\bigcup_{t \in H} t^{-1}A\big)$.

Necessity. Assume that A is piecewise syndetic and pick $H \in \mathcal{P}_f(S)$ such that for all $T \in \mathcal{P}_f(S)$, there exists $x \in S$ such that $T \cdot x \subseteq \bigcup_{t \in H} t^{-1}A$. Then $\{a^{-1}\big(\bigcup_{t \in H} t^{-1}A\big) : a \in S\}$ has the finite intersection property. Pick $q \in \beta S$ such that $\{a^{-1}\big(\bigcup_{t \in H} t^{-1}A\big) : a \in S\} \subseteq q$. Then $S \cdot q \subseteq \overline{\bigcup_{t \in H} t^{-1}A}$. This implies that $(\beta S) \cdot q \subseteq \overline{\bigcup_{t \in H} t^{-1}A}$. We can choose $y \in K(\beta S) \cap (\beta S \cdot q)$. We then have $y \in \overline{t^{-1}A}$ for some $t \in H$ and so $t \cdot y \in \overline{A} \cap K(\beta S)$. □

The algebraic characterization of piecewise syndetic has a straight forward extension to partial semigroups.

3.3 Definition. Let $(S,*)$ be an adequate partial semigroup and let $A \subseteq S$. The set A is piecewise syndetic if and only if $\overline{A} \cap K(\delta S) \neq \emptyset$.

3.4 Lemma. *Let $(S,*)$ be an adequate partial semigroup and let $A \subseteq S$. If A is either thick or syndetic, then A is piecewise syndetic.*

Proof. This is an immediate consequence of Definitions 2.12, 2.19, and 3.3. □

3.5 Theorem. *Let $(S,*)$ be an adequate partial semigroup and let $p \in \delta S$. Then $p \in \overline{K(\delta S)}$ if and only if every $A \in p$ is piecewise syndetic.*

Proof. We have $p \in \overline{K(\delta S)}$ if and only if for each $A \in p$, $\overline{A} \cap K(\delta S) \neq \emptyset$. □

In extending the combinatorial definition of piecewise syndetic to an adequate partial semigroup we recall that the operation $*$ is defined only on the subset $\bigcup_{x\in S}\big(\{x\} \times \varphi(x)\big)$. Therefore, given subsets T and H of S, $h * t * x$ need not be defined for every choice of members t of T and h of H. The references to $\sigma(T)$ and $\sigma(H)$ in the definition that follows guarantee that the operation occurring therein is defined.

3.6 Definition. Let $(S,*)$ be an adequate partial semigroup and let $A \subseteq S$. The set A is *č-piecewise syndetic* if and only if there exists $H \in \mathcal{P}_f(S)$ such that for all $T \in \mathcal{P}_f(S)$ there exists $x \in \sigma(T)$ such that $\big(T \cap \sigma(H)\big) * x \subseteq \bigcup_{t\in H} t^{-1}A$.

In the definitions of č-thick and č-syndetic, the adaptations needed to extend the combinatorial definitions to adequate partial semigroups were fairly obvious. In the case of č-piecewise syndetic we have made one of several possible choices. Theorems 3.8 and 3.12 (especially the latter) suggest that we made the correct choice.

The notions of piecewise syndetic and č-piecewise syndetic are not equivalent, though the notion of piecewise syndetic is stronger than the notion of č-piecewise syndetic. We show that a č-piecewise syndetic set need not be a piecewise syndetic set, again using the adequate partial semigroup from Chapter 1.

3.7 Theorem. *There exist an adequate partial semigroup $(S,*)$ and a subset A of S such that A is č-piecewise syndetic but not piecewise syndetic.*

Proof. Let T be the free semigroup on the generators $\langle y_n\rangle_{n=1}^{\infty}$. Further, let $S = FP(\langle y_n\rangle_{n=1}^{\infty})$, where $\prod_{n\in F} y_n * \prod_{n\in G} y_n$ is defined exactly when $\max F < \min G$.

Let $A = \{\prod_{n \in F} y_n : F \in \mathcal{P}_f(\mathbb{N}) \text{ and } 1 \in F\}$. Then $\delta S \subseteq \varphi(y_1) = \overline{FP(\langle y_n \rangle_{n=2}^{\infty})}$ and $A \cap FP(\langle y_n \rangle_{n=2}^{\infty}) = \emptyset$. In particular $\overline{A} \cap K(\delta S) = \emptyset$, so A is not piecewise syndetic.

To see that A is $\check{c}$-piecewise syndetic, let $H = \{y_1\}$. Let $T \in \mathcal{P}_f(S)$ be given. For each $w \in T$, pick $F_w \in \mathcal{P}_f(\mathbb{N})$ such that $w = \prod_{n \in F_w} y_n$. Let $m = \max \bigcup_{w \in T} F_w$. Then $y_{m+1} \in \sigma(T)$. So if we let $a \in \big(T \cap \sigma(H)\big)$, then $a * y_{m+1}$ is defined and in $\varphi(y_1)$. So $y_1 * a * y_{m+1} \in A$. Therefore $a * y_{m+1} \in y_1^{-1} A = \bigcup_{t \in H} t^{-1} A$.

□

We are able to characterize $\check{c}$-piecewise syndetic sets in terms of the smallest ideal of δS. As with the previous notions, the characterization of $\check{c}$-piecewise syndetic is in terms of the set $\beta S * p$. The reader is reminded that for any $q \in \beta S$, $q * p$ need not be defined unless $p \in \delta S$; and so $\beta S * p$ is not in general a left ideal.

3.8 Theorem. *Let $(S, *)$ be an adequate partial semigroup and let $A \subseteq S$. The set A is*

$\check{c}$-piecewise syndetic if and only if there exists $p \in K(\delta S)$ such that $\overline{A} \cap (\beta S * p) \neq \emptyset$.

Proof. Necessity. Assume that A is $\check{c}$-piecewise syndetic and pick $H \in \mathcal{P}_f(S)$ as guaranteed. For $T \in \mathcal{P}_f(S)$ let $B(T) = \{x \in \sigma(T) : \big(T \cap \sigma(H)\big) * x \subseteq \bigcup_{t \in H} t^{-1} A\}$. Note that $B(T_1 \cup T_2) \subseteq B(T_1) \cap B(T_2)$ and by assumption each $B(T) \neq \emptyset$. So $\{B(T) : T \in \mathcal{P}_f(S)\}$ has the finite intersection property. So pick $p \in \beta S$ such that $\{B(T) : T \in \mathcal{P}_f(S)\} \subseteq p$. Since for all $T \in \mathcal{P}_f(S)$, $B(T) \subseteq \sigma(T)$, we have $p \in \delta S$. Then $\delta S * p$ is a left ideal of δS and so we can pick $q \in K(\delta S)$ such that $q \in \delta S * p$. We claim that $\overline{A} \cap (\beta S * q * p) \neq \emptyset$. It suffices to show that there exists $t \in S$ such that $A \in t * q * p$. Suppose not, then $\bigcup_{t \in H} t^{-1} A \notin q * p$. So $\{s \in S : s^{-1}(\bigcup_{t \in H} t^{-1} A) \notin p\} \in q$. Also, $\sigma(H) \in q$ so pick $s \in \sigma(H)$ such that $s^{-1}(\bigcup_{t \in H} t^{-1} A) \notin p$. Let $T = \{s\}$. Then $B(T) \in p$. Pick $x \in B(T) \backslash \big(s^{-1}(\bigcup_{t \in H} t^{-1} A)\big)$. Then $x \in B(T)$ and $s \in T \cap \sigma(H)$ so $s * x \in \bigcup_{t \in H} t^{-1} A$. This is a contradiction. Thus $\overline{A} \cap (\beta S * q * p) \neq \emptyset$. Since $q * p \in K(\delta S)$, the result follows.

Sufficiency. Pick $p \in K(\delta S)$ such that $\overline{A} \cap (\beta S * p) \neq \emptyset$. So pick $t \in S$ such that $A \in t * p$. Then $t^{-1}A \in p$. Let $B = \{a \in S : a^{-1}(t^{-1}A) \in p\}$. By Theorem 2.28, B is $\check{c}$-syndetic, so pick $H \in \mathcal{P}_f(S)$ such that $\sigma(H) \subseteq \bigcup_{s \in H} s^{-1}B$. Let $G = (t*H) \cup H$. Then $G \in \mathcal{P}_f(S)$. For $T \in \mathcal{P}_f(S)$, we show that there exists $x \in \sigma(T)$ such that $\big(T \cap \sigma(G)\big) * x \subseteq \bigcup_{t \in G} t^{-1}A$. Given $y \in \big(T \cap \sigma(G)\big)$, we have that $y \in \sigma(H)$ so choose $s_y \in H$ such that $s_y * y \in B$. So $(s_y * y)^{-1}(t^{-1}A) \in p$. Also $\sigma(T) \in p$. Pick $x \in \sigma(T) \cap \bigcap_{y \in (T \cap \sigma(G))} (s_y * y)^{-1}(t^{-1}A)$. Then for each $y \in T \cap \sigma(G)$, $s_y * y * x \in t^{-1}A$ and thus $t * s_y * y * x \in A$ and so $y * x \in (t * s_y)^{-1}A$. Thus $\big(T \cap \sigma(G)\big) * x \subseteq \bigcup_{t \in G} t^{-1}A$ and so A is $\check{c}$-piecewise syndetic. □

3.9 Corollary. *Let $(S, *)$ be an adequate partial semigroup and let $A \subseteq S$. If A is either $\check{c}$-thick or $\check{c}$-syndetic, then A is $\check{c}$-piecewise syndetic.*

Proof. If A is $\check{c}$-thick, then by Theorem 2.15 there exists $p \in \delta S$ such that $\beta S * p \subseteq \overline{A}$. Since $\delta S * p$ is a left ideal of δS, we may pick $q \in K(\delta S) \cap (\delta S * p)$. We have that $\beta S * q \subseteq \beta S * p$ and so $\overline{A} \cap (\beta S * q) \neq \emptyset$. (To see that $\beta S * q \subseteq \beta S * p$, pick $r \in \delta S$ such that $q = r * p$. Then for any $t \in \beta S$, $t * q = t * (r * p)$. And by Theorem 1.9, $t * (r * p) = (t * r) * p$.)

If A is $\check{c}$-syndetic, then by Theorem 2.25 for all $p \in \delta S$ one has $\overline{A} \cap (\beta S * p) \neq \emptyset$. □

An immediate consequence of Theorem 3.8 is the following.

3.10 Theorem. *Let $(S, *)$ be an adequate partial semigroup and let $A \subseteq S$ be piecewise syndetic. Then A is $\check{c}$-piecewise syndetic.*

Proof. Pick $p \in K(\delta S)$ such that $A \in p$. Let L be a minimal left ideal of δS containing p. Then $L * p$ is a left ideal and $L * p \subseteq L$. So $L * p = L$. Since $p \in L * p$ we have $\overline{A} \cap (\beta S * p) \neq \emptyset$. Thus, by Theorem 3.8 A is $\check{c}$-piecewise syndetic. □

We mentioned in the introduction to this chapter that in a semigroup, the piecewise syndetic sets are precisely those sets that are the intersection of a thick set with a syndetic set. Since in an adequate partial semigroup the notions of $\check{c}$-piecewise syndetic and piecewise syndetic are not equivalent, we are interested in determining whether they have corresponding relationships with respect to thick (respectively $\check{c}$-thick) and syndetic (respectively $\check{c}$-syndetic) sets.

We first provide the proof of the known result for a semigroup; this is taken from [3].

3.11 Theorem. *Let $(S, \cdot)$ be a semigroup and suppose $A \subseteq S$. Then A is piecewise syndetic if and only if there exist a syndetic set B and a thick set C such that $A = B \cap C$.*

Proof. Necessity. Pick $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t \in H} t^{-1}A$ is thick. Let $C = A \cup \bigcup_{t \in H} t^{-1}A$ and let $B = A \cup (S \backslash C)$. Then trivially C is thick and $A = B \cap C$. Thus it suffices to show that B is syndetic. Suppose not. Then $S \backslash B$ is thick and $S \backslash B = C \backslash A \subseteq \bigcup_{t \in H} t^{-1}A$.

Recall that any semigroup is an adequate partial semigroup and that the notions of thick and $\check{c}$-thick coincide for semigroups. Pick by Lemma 2.18 some $x \in S \backslash B$ such that $Hx \subseteq S \backslash B$. Then for some $t \in H, tx \in A$ so $tx \in B$, a contradiction.

Sufficiency. Now assume that $A = B \cap C$ where B is syndetic and C is thick. Pick $H \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in H} t^{-1}B$. Let $F \in \mathcal{P}_f(S)$ be given and pick x such that $HFx \subseteq C$. We claim that $Fx \subseteq \bigcup_{t \in H} t^{-1}(B \cap C)$. To see this, let $y \in F$ and pick $t \in H$ such that $yx \in t^{-1}B$. Then $tyx \in B \cap C$. □

There is an identical relationship between $\check{c}$-piecewise syndetic sets, $\check{c}$-thick and $\check{c}$-syndetic sets.

3.12 Theorem. *Let S be an adequate partial semigroup and suppose $A \subseteq S$. The set A is $\check{c}$-piecewise syndetic if and only if there exist a $\check{c}$-syndetic set B, and a $\check{c}$-thick set C, such that $A = B \cap C$.*

Proof. Sufficiency. Let B be a $\check{c}$-syndetic subset of S and C a $\check{c}$-thick subset of S with $A = B \cap C$. Pick by Theorem 2.15 $p \in \delta S$ such that $\beta S * p \subseteq \overline{C}$, and pick $q \in (\delta S * p) \cap \big(K(\delta S)\big)$. As in the proof of Corollary 3.9 we have that $\beta S * q \subseteq \beta S * p$ and thus $\beta S * q \subseteq \beta S * p \subseteq \overline{C}$. Since B is $\check{c}$-syndetic, $(\beta S * q) \cap \overline{B} \neq \emptyset$ by Theorem 2.25. Therefore $(\beta S * q) \cap \overline{A} \neq \emptyset$. So by Theorem 3.8, A is $\check{c}$-piecewise syndetic.

Necessity. Assume A is $\check{c}$-piecewise syndetic, and pick $H \in \mathcal{P}_f(S)$ such that for all $T \in \mathcal{P}_f(S)$ there exists $x_T \in \sigma(T)$ such that $\big(T \cap \sigma(H)\big) * x_T \subseteq \bigcup_{t \in H} t^{-1}A$.

Let $C = A \cup \bigcup_{t \in H} t^{-1}A \cup \{y * x_T : T \in \mathcal{P}_f(S) \text{ and } y \in T \backslash \sigma(H)\}$ and let $B = A \cup (S \backslash C)$. Then $A = B \cap C$. We claim that B is $\check{c}$-syndetic and C is $\check{c}$-thick. To see that C is $\check{c}$-thick, let $F \in \mathcal{P}_f(S)$ so that $(F \cap \sigma(H)) * x_F \subseteq \bigcup_{t \in H} t^{-1}A$. Let $w \in F$. If $w \in \sigma(H)$, then $w * x_F \in \bigcup_{t \in H} t^{-1}A \subseteq C$. And if $w \notin \sigma(H)$, $w * x_F \in \{y * x_T : T \in \mathcal{P}_f(S) \text{ and } y \in T \backslash \sigma(H)\} \subseteq C$. So C is $\check{c}$-thick.

To see that B is $\check{c}$-syndetic, suppose not. Then by Theorem 2.24 $S \backslash B$ is $\check{c}$-thick. Now $S \backslash B = C \backslash A \subseteq \bigcup_{t \in H} t^{-1}A \cup \{y * x_T : T \in \mathcal{P}_f(S) \text{ and } y \in T \backslash \sigma(H)\}$. By Lemma 2.18, we can pick $z \in (S \backslash B) \cap \sigma(H)$ such that $H * z \subseteq S \backslash B$. We claim that $z \in \{y * x_T : T \in \mathcal{P}_f(S) \text{ and } y \in T \backslash \sigma(H)\}$. Otherwise $z \in \bigcup_{t \in H} t^{-1}A$, in which case we can pick $t \in H$ such that $t * z \in A$ while $t * z \in S \backslash B = C \backslash A$. This is a contradiction. So $z \in \sigma(H)$ and $z = y * x_T$ for some $T \in \mathcal{P}_f(S)$ and some $y \in T \backslash \sigma(H)$. Since $y \notin \sigma(H)$, pick $t \in H$ such that $y \notin \varphi(t)$. But $t * z = t * (y * x_T)$ and so $t * y$ is defined, a contradiction. □

For the algebraic notion, we are able to establish only the reverse implication. We do not know whether every piecewise syndetic set is the intersection of a syndetic set with a thick set.

3.13 Theorem. *Let S be an adequate partial semigroup and suppose $A \subseteq S$. If there exists a syndetic set B and a thick set C such that $A = B \cap C$ then A is piecewise syndetic.*

Proof. Let C be a thick subset of S and B a syndetic subset of S with $A = B \cap C$.

Pick $p \in \delta S$ such that $\delta S * p \subseteq \overline{C}$. Since B is syndetic, $\overline{B} \cap (\delta S * p) \neq \emptyset$ and thus $\overline{A} \cap (\delta S * p) \neq \emptyset$. □

The following theorem shows some of the interrelationships between syndetic sets and piecewise syndetic sets for an adequate partial semigroup.

3.14 Theorem. *Let S be an adequate partial semigroup and suppose $A \subseteq S$. The following statements are equivalent:*

(a) $\overline{A} \cap K(\delta S) \neq \emptyset$.

(b) There exists $p \in K(\delta S)$ such that $\{x \in S : x^{-1}A \in p\}$ is syndetic.

(c) There exists $p \in \delta S$ such that $\{x \in S : x^{-1}A \in p\}$ is syndetic.

(d) There exists $p \in \delta S$ such that $\{x \in S : x^{-1}A \in p\}$ is piecewise syndetic.

Proof. $(a) \Rightarrow (b)$. Pick $p \in K(\delta S) \cap \overline{A}$. Then by Theorem 2.28, $\{x \in S : x^{-1}A \in p\}$ is syndetic.

$(b) \Rightarrow (c)$. Trivial.

$(c) \Rightarrow (d)$. Lemma 3.4.

$(d) \Rightarrow (a)$. Pick p as guaranteed. Let $B = \{x \in S : x^{-1}A \in p\}$. Since B is piecewise syndetic, pick $q \in K(\delta S)$ such that $B \in q$. So $\{x \in S : x^{-1}A \in p\} \in q$ so $A \in q * p$. Therefore $\overline{A} \cap K(\delta S) \neq \emptyset$. □

Piecewise syndeticity and č-piecewise syndeticity are partition regular properties for adequate partial semigroups.

3.15 Theorem. *In an adequate partial semigroup, the piecewise syndetic property is partition regular.*

Proof. Assume that $A \cup B$ is piecewise syndetic. Then $(\overline{A} \cup \overline{B}) \cap K(\delta S) = (\overline{A \cup B}) \cap K(\delta S) \neq \emptyset$. So either $\overline{A} \cap K(\delta S) \neq \emptyset$ or $\overline{B} \cap K(\delta S) \neq \emptyset$. That is, either A is piecewise syndetic or B is piecewise syndetic. □

3.16 Theorem. *In an adequate partial semigroup, the č-piecewise syndetic property is partition regular.*

Proof. Assume that $A \cup B$ is č-piecewise syndetic and pick, by Theorem 3.8, some $p \in K(\delta S)$ and some $q \in \beta S$ such that $A \cup B \in q * p$. Then either $A \in q * p$ or $B \in q * p$. That is, either A is č-piecewise syndetic or A is č-piecewise syndetic. $\square$

CHAPTER IV

IP and CENTRAL SETS

In this chapter we look at the significance of idempotents in an adequate partial semigroup. As in a semigroup, an element x in a partial semigroup $(S, *)$ is an *idempotent* if and only if $x = x * x$. Idempotents in a semigroup are of particular interest when considering the algebraic structure of βS. One popular result is the Finite Sums Theorem which establishes an intimate relationship between finite sums in a semigroup S and idempotents of βS. In topological dynamics, the notion of an *IP set* in the semigroup $(\mathbb{N}, +)$ is defined to be a set which can be written as $FS(\langle x_n \rangle_{n=1}^{\infty})$ for some sequence $\langle x_n \rangle_{n=1}^{\infty}$. For a semigroup written multiplicatively, the corresponding requirement would be to have a set which can be written as $FP(\langle x_n \rangle_{n=1}^{\infty})$. Additionally, since notions of largeness ought to be closed under passage to supersets, one can simply ask that a set contain some $FP(\langle x_n \rangle_{n=1}^{\infty})$. This notion is found to have an equivalent characterization in terms of the algebra of βS for an arbitrary semigroup S. In the context of compact right topological semigroups IP sets are characterized in terms of idempotents.

Central sets in semigroups are well researched in the current literature. Central sets are known to have very rich combinatorial structure as described by the "Central Sets Theorem" [6, Proposition 8.21] or [9, Theorem 14.11]. Central sets also originate in topological dynamics and were defined for subsets of $\mathbb{N}$ by Furstenberg. Algebraically, central sets are sets that are elements of minimal idempotents. Formally, a subset A of a semigroup S is central if and only if it is a member of an idempotent in the smallest ideal of βS. In [5] an analog of the Central Sets Theroem for adequate partial semigroups was presented. Some of the material from [5] will be provided here also.

As in the context of compact right topological semigroups, we attempt to characterize IP and central sets in adequate partial semigroups. Central sets in an adequate partial semigroup are characterized using piecewise syndetic sets. So we will reference to the material covered in the previous chapter when constructing this characterization. As both notions, IP and central, are intimately related to idempotents, we expect that IP and central sets are also related. This explains our choice of title for this chapter.

We begin, as before, by presenting the formal definitions and results of IP followed by central sets in a semigroup. Our extension of IP to partial semigroups is dual, as with all our preceding notions. However, our semigroup definition of a central set is in fact its algebraic characterization in βS, therefore when we extend central to partial semigroups we do so algebraically only. (There is a combinatorial characterization of central sets [9, Section 14.5]. However, it is quite complicated and does not have an obvious extension to partial semigroups.)

4.1 Definition. Let S be a semigroup. A subset A of S is an *IP set* if and only if there is a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in S such that $FP(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A$.

4.2 Theorem. *Let S be a semigroup and let $A \subseteq S$. Then A is an IP set if and only if there is some idempotent $p \in \beta S$ such that $A \in p$.*

Proof. Necessity. Since A is IP we can pick a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in S such that $FP(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A$. Let $T = \bigcap_{m=1}^{\infty} \overline{FP(\langle x_n\rangle_{n=m}^{\infty})}$. We show that T is a semigroup. For then $E(T) \neq \emptyset$, so the conclusion is satisfied. To see that T is a semigroup, we use [9, Theorem 4.20]. Notice that trivially $\{FP(\langle x_n\rangle_{n=m}^{\infty}) : m \in \mathbb{N}\}$ has the finite intersection property. Let $m \in \mathbb{N}$ and let $s \in FP(\langle x_n\rangle_{n=m}^{\infty})$be given. Pick $F \in \mathcal{P}_f(\mathbb{N})$ with $\min F \geq m$ such that $s = \prod_{n\in F} x_n$. Let $k = \max F + 1$. To see that $s \cdot FP(\langle x_n\rangle_{n=k}^{\infty}) \subseteq FP(\langle x_n\rangle_{n=m}^{\infty})$, let $t \in FP(\langle x_n\rangle_{n=k}^{\infty})$be given and pick $G \in \mathcal{P}_f(\mathbb{N})$ with $\min G \geq k$ such that $t = \prod_{n\in G} x_n$. Then $\max F < \min G$ so $st = \prod_{n\in F\cup G} x_n \in FP(\langle x_n\rangle_{n=m}^{\infty})$. Therefore T is a semigroup and so contains an idempotent.

Sufficiency. Pick $p \in \beta S$ with $p \cdot p = p$, such that $A \in p$. Let $A_1 = A$ and let $B_1 = \{x \in S : x^{-1}A_1 \in p\}$. $A_1 \in p \cdot p$ (since $p = p \cdot p$), so $\{x \in S : x^{-1}A_1\} \in p$. So $B_1 \in p$. Pick $x_1 \in B_1 \cap A_1$ and let $A_2 = A_1 \cap (x^{-1}A_1)$. So $A_2 \in p$. Inductively, given $A_n \in p$, let $B_n = \{x \in S : x^{-1}A_n \in p\}$. Then $B_n \in p$, so pick $x_n \in B_n \cap A_n$, and let $A_{n+1} = A_n \cap (x_n^{-1}A_n)$. We have produced a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S.

We show that if $F \in \mathcal{P}_f(\mathbb{N})$ and $m = \min F$ then $\prod_{n \in F} x_n \in A_m$. To see this, if $|F| = 1$, then $\prod_{n \in F} x_n = x_m \in A_m$. If $|F| > 1$, let $G = F \backslash \{m\}$, and let $k = \min G$. Since $k > m$, $A_k \subseteq A_{m+1}$. Then by the induction hypothesis, $\prod_{n \in G} x_n \in A_k \subseteq A_{m+1} \subseteq x_m^{-1}A_m$. So $\prod_{n \in F} x_n = x_m \cdot \prod_{n \in G} x_n \in A_m$. Therefore A is IP. □

Being itself a compact right topological semigroup, δS contains idempotents. Therefore we are able to easily extend the algebraic characterization of IP to partial semigroups.

4.3 Definition. Let S be an adequate partial semigroup with $A \subseteq S$. A is IP if and only if there exists $p = p * p \in \delta S$ such that $A \in p$.

For an adequate partial semigroup S and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S, the finite products $FP(\langle x_n \rangle_{n=1}^{\infty})$ need not be defined for each $n \in \mathbb{N}$. Thus to extend the combinatorial definition of IP to the partial case requires the following slight modification of Definition 4.1.

4.4 Definition. Let S be an adequate partial semigroup with $A \subseteq S$. A is $\check{c}$-IP if and only if there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that for all $F \in \mathcal{P}_f(\mathbb{N})$, $\prod_{n \in F} x_n$ is defined and $\prod_{n \in F} x_n \in A$.

As would seem likely, the notions IP and $\check{c}$-IP are not equivalent. In the example that follows one should notice that as defined, a $\check{c}$-IP set fails to be IP precisely when for some $H \in \mathcal{P}_f(S)$, $FP(\langle x_n \rangle_{n=1}^{\infty}) \cap \sigma(H) = \emptyset$. As we shall see in Theorem 4.8, this leads to a necessary and sufficient condition for equivalence of the two notions.

4.5 Theorem. *There exists an adequate partial semigroup $(S, *)$ and a subset A of S such that A is č-IP but not IP.*

Proof. Let $S = \{A \subseteq \mathbb{N} : \; A \neq \emptyset \text{ and } |A \backslash 2\mathbb{N}| < \omega\}$. So S is the collection of nonempty subsets of $\mathbb{N}$ with finitely many odd numbers. Define $\uplus$ on S such that

$$A \uplus B = \begin{cases} A \cup B & \text{if } A \cap B = \emptyset \\ \text{undefined} & \text{if } A \cap B \neq \emptyset \end{cases}$$

Then $(S, \uplus)$ is an adequate partial semigroup.

To see this, let $\mathcal{H} = \{A_1, A_2, \ldots, A_n\} \subseteq S$. Then $|\bigcup_{i=1}^{n} A_i \backslash 2\mathbb{N}| < \omega$. So pick $x \in \mathbb{N} \backslash \bigcup_{i=1}^{n} A_i$. Then $\{x\} \in \varphi(A_i)$ for $i \in \{1, 2, \ldots, n\}$. So $\{x\} \in \bigcap_{i=1}^{n} \varphi(A_i) = \sigma(\mathcal{H})$. Therefore $\sigma(\mathcal{H}) \neq \emptyset$ so $(S, \uplus)$ is adequate.

Let $\mathcal{A} = \mathcal{P}_f(2\mathbb{N})$. We claim that $\mathcal{A}$ is č-IP but not IP. $\mathcal{A}$ is č-IP since $\mathcal{A} = FP(\langle\{2n\}\rangle_{n=1}^{\infty})$. To see $\mathcal{A}$ is not IP, we show that there is no $p \in \delta S$, idempotent or otherwise, with $\mathcal{A} \in p$. Suppose there exists $p \in \delta S$ such that $\mathcal{A} = \mathcal{P}_f(2\mathbb{N}) \in p$. Notice that $2\mathbb{N} \in S$, and $\varphi(2\mathbb{N}) = \mathcal{P}_f(2\mathbb{N} - 1) \in p$. However, $\mathcal{P}_f(2\mathbb{N}) \cap \varphi(2\mathbb{N}) = \emptyset$. This is a contradiction. Thus $\mathcal{A}$ is not IP. □

The next theorem shows that if a set A is IP, then there is a sequence whose finite products are defined and contained in A. That is, IP implies č-IP. The proof is verbatim the sufficiency part of the proof of Theorem 4.2. This is the case because as in the earlier proof, we can see for instance why $x_3 * x_4 * x_6 * x_8 \in A$ since $x_8 \in A_8 \subseteq A_7 \subseteq x_6^{-1} A_6$, therefore $x_6 * x_8$ is defined and in A_6. Likewise, $A_6 \subseteq A_5 \subseteq x_4^{-1} A_4$, so $x_4 * x_6 * x_8$ is also defined and in A_4. While $A_4 \subseteq x_3^{-1} A_3$ so $x_3 * x_4 * x_6 * x_8$ is defined and in $A_3 \subseteq A_2 \subseteq A_1 = A$.

4.6 Theorem. *Let $(S, *)$ be an adequate partial semigroup and suppose $A \subseteq S$. If A is IP, then A is č-IP.*

Proof. Pick $p \in \delta S$ with $p * p = p$, such that $A \in p$. Let $A_1 = A$ and let $B_1 = \{x \in S : x^{-1} A_1 \in p\}$. $A_1 \in p * p$ (since $p = p * p$), so $\{x \in S : x^{-1} A_1\} \in p$.

So $B_1 \in p$. Pick $x_1 \in B_1 \cap A_1$ and let $A_2 = A_1 \cap (x^{-1}A_1)$. So $A_2 \in p$. Inductively, given $A_n \in p$, let $B_n = \{x \in S : x^{-1}A_n \in p\}$. Then $B_n \in p$, so pick $x_n \in B_n \cap A_n$, and let $A_{n+1} = A_n \cap (x_n^{-1}A_n)$. We have produced a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S.

We show that if $F \in \mathcal{P}_f(\mathbb{N})$ and $m = \min F$ then $\prod_{n\in F} x_n \in A_m$. To see this, if $|F| = 1$, then $\prod_{n\in F} x_n = x_m \in A_m$. If $|F| > 1$, let $G = F\backslash\{m\}$, and let $k = \min G$. Since $k > m$, $A_k \subseteq A_{m+1}$. Then by the induction hypothesis, $\prod_{n\in G} x_n \in A_k \subseteq A_{m+1} \subseteq x_m^{-1}A_m$. So $\prod_{n\in F} x_n = x_m * \prod_{n\in G} x_n \in A_m$. Therefore A is $\check{c}$-IP. (Notice that the fact that the product $x_m * \prod_{n\in G} x_n \in A_m$ is defined is part of the requirement for $\prod_{n\in G} x_n \in x_m^{-1}A_m$.)

As was the case for a semigroup, the last result may also be proved in the following way.

Alternate Proof:

Let $p = p * p \in \delta S$ and $A \in p$. Let $A^\star = \{s \in A : s^{-1}A \in p\}$. Pick $x_1 \in A^\star$. Let $n \in \mathbb{N}$, and assume we have chosen $\langle x_n \rangle_{n=1}^{\infty}$ such that $FP(\langle x_t \rangle_{t=1}^{n}) \subseteq A^\star$. Let $E = FP(\langle x_t \rangle_{t=1}^{n})$. Then E is finite, and by Lemma 1.21 we have that for each $a \in E$, $a^{-1}A^\star \in p$. So $\bigcap_{a\in E} a^{-1}A^\star \in p$. Pick $x_{n+1} \in A^\star \cap \bigcap_{a\in E} a^{-1}A^\star$. Then $FP(\langle x_t \rangle_{t=1}^{n+1}) \subseteq A^\star \subseteq A$. So A is IP. □

As was mentioned earlier in this chapter, though č-IP does not imply IP, we are able to give conditions for which the two notions are equivalent. Theorem 4.8 below gives precise conditions under which a č-IP set can be guaranteed to be an IP set. This is particularly interesting since, to this point, we have not been able to do this for any other notion. The lemma that follows is needed for the proof of Theorem 4.8.

4.7 Lemma. *Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence such that $\prod_{n\in F} x_n$ is defined for all $F \in \mathcal{P}_f(\mathbb{N})$. The following are equivalent:*

(a) $\{FP(\langle x_n \rangle_{n=m}^{\infty}) \cap \varphi(y) : m \in \mathbb{N}$ and $y \in S\}$ has the finite intersection property.

(b) $\bigcap_{m=1}^{\infty} \overline{FP(\langle x_n \rangle_{n=m}^{\infty})} \cap \delta S$ is a semigroup.

Proof. (a) $\Rightarrow$ (b). Let $T = \bigcap_{m=1}^{\infty} \overline{FP(\langle x_m \rangle_{n=m}^{\infty})} \cap \delta S = \bigcap_{m=1}^{\infty} \overline{FP(\langle x_m \rangle_{n=m}^{\infty})} \cap \bigcap_{H \in \mathcal{P}_f(S)} \overline{\sigma(H)}$. We have that $T \neq \emptyset$ by assumption. Let $p, q \in T$. To see that $p{*}q \in T$, let $m \in \mathbb{N}$, let $H \in \mathcal{P}_f(S)$, and let $A = FP(\langle x_n \rangle_{n=m}^{\infty}) \cap \sigma(H)$. We show that $A \subseteq \{s \in S : s^{-1}A \in q\}$ so that $A \in p{*}q$. To see this, let $s \in A$, and pick $F \in \mathcal{P}_f(\mathbb{N})$ such that $s = \prod_{n \in F} x_n$. Let $k = \max F + 1$, and let $L = H * s$. (Notice that since $s \in \sigma(H)$, $y{*}s$ is defined for all $y \in H$.) We claim that $FP(\langle x_n \rangle_{n=k}^{\infty}) \cap \sigma(L) \subseteq s^{-1}A$, so that $s^{-1}A \in q$. To see this, let $t \in FP(\langle x_n \rangle_{n=k}^{\infty}) \cap \sigma(L)$. One has immediately that $s * t \in FP(\langle x_n \rangle_{n=m}^{\infty})$. To see that $s * t \in \sigma(H)$, let $h \in H$. Then $h * s \in L$, so $(h * s) * t$ is defined. So $h * (s * t)$ is also defined. Therefore $(s * t) \in \sigma(H)$. Thus, $t \in s^{-1}A$.

(b) $\Rightarrow$ (a). Since $T = \bigcap_{m=1}^{\infty} \overline{FP(\langle x_m \rangle_{n=m}^{\infty})} \cap \delta S$ is a semigroup, $T \neq \emptyset$. Given $p \in T$, $\{FP(\langle x_n \rangle_{n=m}^{\infty}) \cap \sigma(H) : m \in \mathbb{N} \text{ and } H \in \mathcal{P}_f(S)\} \subseteq p$. $\square$

4.8 Theorem. *Let $(S, *)$ be an adequate partial semigroup. The following are equivalent:*

(a) For all $A \subseteq S$, A is č-IP if and only if A is IP.

(b) Whenever $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in S such that $\prod_{n \in F} x_n$ is defined for all $F \in \mathcal{P}_f(\mathbb{N})$ and $H \in \mathcal{P}_f(S)$, $FP(\langle x_n \rangle_{n=1}^{\infty}) \cap \sigma(H) \neq \emptyset$.

(c) Whenever $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in S such that $\prod_{n \in F} x_n$ is defined for all $F \in \mathcal{P}_f(\mathbb{N})$, $\{FP(\langle x_n \rangle_{n=m}^{\infty}) \cap \varphi(y) : m \in \mathbb{N}$ and $y \in S\}$ has the finite intersection property.

Proof. (a) $\Rightarrow$ (b). Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in S such that for all $F \in \mathcal{P}_f(\mathbb{N})$, $\prod_{n \in F} x_n$ is defined and let $A = FP(\langle x_n \rangle_{n=1}^{\infty})$. Let $H \in \mathcal{P}_f(S)$. Pick $p \in \delta S$ such that $p = p{*}p$ and $A \in p$. Then $\sigma(H) \in p$ since $p \in \delta S$. So $FP(\langle x_n \rangle_{n=1}^{\infty}) \cap \sigma(H) \neq \emptyset$.

(b) $\Rightarrow$ (c). Let $F \in \mathcal{P}_f(\mathbb{N})$ and let $H \in \mathcal{P}_f(S)$. Let $k = \max F$. Then (b) applied to the sequence $\langle x_n \rangle_{n=k}^{\infty}$ says that

$$\emptyset \neq FP(\langle x_n \rangle_{n=k}^{\infty}) \cap \sigma(H) \subseteq \bigcap_{m \in F} FP(\langle x_n \rangle_{n=m}^{\infty}) \cap \bigcap_{y \in H} \varphi(y).$$

(c) $\Rightarrow$ (a). Let A be $\check{c}$-IP and let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in S such that $\prod_{n \in F} x_n$ is defined for all $F \in \mathcal{P}_f(\mathbb{N})$ and $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$. Then by Lemma 4.7, $T = \bigcap_{m=1}^{\infty} \overline{FP(\langle x_n \rangle_{n=m}^{\infty})} \cap \delta S$ is a semigroup. So pick p, an idempotent in T. Then $FP(\langle x_n \rangle_{n=1}^{\infty}) \in p$ and $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$. So $A \in p$. Therefore A is IP. By Theorem 4.6 we know that IP implies $\check{c}$-IP. □

We now direct our attention to the notion of a central set in an adequate partial semigroup. As noted earlier, central sets have rich combinatorial structure. One combinatorial property of central sets in compact right topological semigroups is that they contain arbitrarily long arithmetic progressions. Another is that every central set contains an IP set.

Unlike the other notions we have considered, we will define central sets only in terms of their algebraic characterization in βS. Thus in the partial semigroup case we will only have one notion of central. We begin by reviewing the meaning of central in a semigroup. Recall that an idempotent in a compact right topological semigroup is minimal if and only if it is a member of the smallest ideal.

4.9 Definition. Let $(S, \cdot)$ be a discrete semigroup and let $A \subseteq S$. Then A is *central* if and only if there is some minimal idempotent p in βS such that $A \in p$.

In an adequate partial semigroup the definition of central is the same.

4.10 Definition. Let $(S, *)$ be an adequate partial semigroup and let $A \subseteq S$. Then A is central if and only if there is some minimal idempotent p in δS such that $A \in p$.

In a semigroup there is an intimate relationship between central and piecewise syndetic sets. And since by Theorem 3.11 every piecewise syndetic set is the intersection of a thick set with a syndetic set, central sets are also related to thick and syndetic sets.

4.11 Theorem. *Let S be an infinite semigroup and let $A \subseteq S$. The following statements are equivalent.*

(a) The set A is piecewise syndetic.

(b) The set $\{x \in S : x^{-1}A$ is central $\}$ is syndetic.

(c) There is some $x \in S$ such that $x^{-1}A$ is central.

Proof. (a) $\Rightarrow$ (b). Pick $p \in K(\beta S)$ with $A \in p$. Since $K(\beta S)$ is the union of all minimal left ideals of βS, we can pick a minimal left ideal L of βS with $p \in L$ and pick an idempotent $e \in L$. Then $p = p \cdot e$ so pick $y \in S$ such that $y^{-1}A \in e$.

The set $B = \{z \in S : z^{-1}(y^{-1}A) \in e\}$ is syndetic by Theorem 2.28. So pick finite $G \subseteq S$ such that $S = \bigcup_{t \in G} t^{-1}B$. Let $D = \{x \in S : x^{-1}A \text{ is central}\}$. We claim that $S = \bigcup_{t \in (y \cdot G)} t^{-1}D$. To see this, let $x \in S$ be given and pick $t \in G$ such that $t \cdot x \in B$. Then $(t \cdot x)^{-1}(y^{-1}A) \in e$ so $(t \cdot x)^{-1}(y^{-1}A)$ is central. But $(t \cdot x)^{-1}(y^{-1}A) = (y \cdot t \cdot x)^{-1}A$. Thus $y \cdot t \cdot x \in D$ so $x \in (y \cdot t)^{-1}D$ as required.

(b) $\Rightarrow$ (c). Trivial.

(c)$\Rightarrow$ (a). Pick $x \in S$ such that $x^{-1}A$ is central and pick an idempotent $p \in K(\beta S)$ such that $x^{-1}A \in p$. Then $A \in x \cdot p$ and $x \cdot p \in K(\beta S)$ so by [9, Theorem 4.40], A is piecewise syndetic. $\square$

In the partial case we have not quite the same situation.

4.12 Theorem. *Let $(S, *)$ be an adequate partial semigroup and let $A \subseteq S$. Statements (a) and (b) are equivalent and imply statements (c) and (d).*

(a) The set A is piecewise syndetic.

(b) There exists $e = e * e \in K(\delta S)$ such that $\{t \in S : t^{-1}A \in e\}$ is piecewise syndetic.

(c) The set $\{t \in S : t^{-1}A$ is central $\}$ is piecewise syndetic.

(d) There exists $e = e * e \in K(\delta S)$ such that $\{t \in S : t^{-1}A \in e\}$ is č-syndetic.

Proof. (a) $\Rightarrow$ (b). Pick $p \in K(\delta S)$ such that $A \in p$. Also pick a minimal left ideal L of S such that $p \in L$. At the same time pick a minimal idempotent $e \in L$. Then $p = p * e$, so $A \in p * e$, that is $\{t \in S : t^{-1}A \in e\} \in p$. So $\{t \in S : t^{-1}A \in e\}$ is piecewise syndetic.

(b) $\Rightarrow$ (a). Pick $p \in K(\delta S)$ such that $\{t \in S : t^{-1}A \in e\} \in p$. Then $A \in p * e$ so A is piecewise syndetic.

(a) $\Rightarrow$ (d). Pick $p \in K(\delta S)$ such that $A \in p$. Pick a minmal left ideal L of S with $p \in L$, and at the same time pick a minimal idempotent $e \in L$. Then $p = p * e$, so pick $y \in S$ such tha $y^{-1}A \in e$. Then by Theorem 2.28, $B = \{x \in S : x^{-1}(y^{-1}A) \in e\}$ is $\check{c}$-syndetic. So pick finite $G \subseteq S$ such that $\sigma(G) \subseteq \bigcup_{t \in G} t^{-1}B$. Let $D = \{a \in S : \ a^{-1}A \in e\}$. Let $H = G \cup y * \big(G \cap \varphi(y)\big)$. We claim that $\sigma(H) \subseteq \bigcup_{t \in H} t^{-1}D$. To see this let $w \in \sigma(H)$. Then $w \in \sigma(G)$. So pick $s \in G$ such that $w \in s^{-1}B$. Then $s * w \in B$, and so $(s * w)^{-1}(y^{-1}A) \in e$. That is, $((y * s) * w)^{-1}A \in e$. So $(y * s) * w \in D$ and therefore $w \in (y * s)^{-1}D$. We also have that $y * s \in y * \big(G \cap \varphi(y)\big) \subseteq H$.

(b) $\Rightarrow$ (c). Pick a minimal idempotent e as guaranteed. If $t^{-1}A \in e$, then $t^{-1}A$ is central. $\square$

CHAPTER V

The * Properties

Given any class $\mathcal{R}$ of subsets of a set S, we define the class $\mathcal{R}^*$ to be the class of subsets that meet every member of $\mathcal{R}$ nontrivially. If P is a property and $\mathcal{R} = \{A \subseteq S : \ A \text{ has property P}\}$, then A has property P^* if and only if $A \in \mathcal{R}^*$. Thus if P and Q are properties of subsets of S and we define the classes $\mathcal{R} = \{A : \ A \text{ has property P}\}$, and $\mathcal{S} = \{B : \ B \text{ has property Q}\}$ the statements "$P \Rightarrow Q$" and "$\mathcal{R} \subseteq \mathcal{S}$" are equivalent. We state the following theorem in terms of classes. The corresponding assertion using properties will also be used.

The following facts will be useful for our discussion. When we say that $\mathcal{R} \subseteq \mathcal{P}_f(S)$ is closed under supersets we mean that for all $A \in \mathcal{R}$, and for all $B \subseteq S$, if $A \subseteq B$ then $B \in \mathcal{R}$.

5.1 Theorem. *Let S be a set and let $\mathcal{R} \subseteq \mathcal{P}(S)$ such that $\mathcal{R}$ is closed under supersets. Then*

(a) For all $A \subseteq S$, $A \in \mathcal{R}^*$ if and only if $S \backslash A \notin \mathcal{R}$.

(b) $\left(\mathcal{R}^*\right)^* = \mathcal{R}$.

Proof. (a) Necessity. Let $A \in \mathcal{R}^*$. Then if $B \in \mathcal{R}$, $A \cap B \neq \emptyset$. So $S \backslash A \notin \mathcal{R}$.

Sufficiency. Let $B \in \mathcal{R}$. Suppose that $A \cap B = \emptyset$, then $B \subseteq S \backslash A$. Since $\mathcal{R}$ is closed under supersets, $S \backslash A \in \mathcal{R}$. But $A \cap (S \backslash A) = \emptyset$. This is a contradiction.

(b) Let $A \in \left(\mathcal{R}^*\right)^*$. Then by part (a), $S \backslash A \notin \mathcal{R}^*$. Using the same argument $A \in \mathcal{R}$.

Now let $A \in \mathcal{R}$ and suppose that $A \notin \left(\mathcal{R}^*\right)^*$. By part (a) $S \backslash A \in \mathcal{R}^*$, so $A \notin \mathcal{R}$. This is a contradiction. □

5.2 Theorem. *Let S be a set and let $\mathcal{S}$ and $\mathcal{R}$ be subsets of $\mathcal{P}(S)$ which are closed under supersets. Then $\mathcal{R} \subseteq \mathcal{S}$ if and only if $\mathcal{S}^* \subseteq \mathcal{R}^*$.*

Proof. Necessity. Let $A \in \mathcal{S}^*$. Then by Theorem 5.1, $S \backslash A \notin \mathcal{S}$. By assumption then, $S \backslash A \notin \mathcal{R}$. Applying Theorem 5.1 again, we get that $A \in \mathcal{R}^*$.

Sufficiency. Let $A \in \mathcal{R}$. Then by Theorem 5.1, $S \backslash A \notin \mathcal{R}^*$. So by our assumption $S \backslash A \notin \mathcal{S}^*$. Therefore by Theorem 5.1, $A \in \mathcal{S}$. □

In a semigroup *IP** sets, that is, sets which intersect $FP(\langle x_n \rangle_{n=1}^{\infty})$ for every sequence $\langle x_n \rangle_{n=1}^{\infty}$, are known to have rich combinatorial structure. This is of particular significance in the semigroups $(\mathbb{N}, +)$ and $(\mathbb{N}, \cdot)$, [9], [6]. And *central** sets, which are characterized as sets contained in every minimal idempotent, have applications to image partition regular matrices, [9].

For partial semigroups we are similarly interested in the dual of IP and central sets. In this chapter we will develop the classes analogous to the class of IP* and central* sets for partial semigroups. Notice that, based on our discussion of thick and syndetic sets in Chapter 2, we have already seen some * properties. So for an adequate partial semigroup we saw that a syndetic* (respectively č-syndetic*) set is a thick (respectively č-thick) set and vice-versa.

In a semigroup we have the following characterization of IP* sets.

5.3 Theorem. *Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. Then the following statements are equivalent:*

(a) A is an IP set in S.*

(b) A is a member of every idempotent of βS.

(c) $A \cap B$ is an IP set for every IP subset B of S.

Proof. [9, Theorem 16.6].

Since the notion of IP, as we saw in the last chapter, takes on two nonequivalent meanings in an adequate partial semigroup, we have correspondingly the notions of *IP** and *č-IP**.

The algebraic characterization of IP* sets in a semigroup, (Theorem 5.3), applies almost verbatim to partial semigroups.

5.4 Theorem. *Let $(S,*)$ be an adequate partial semigroup and let $A \subseteq S$. The following statements are equivalent:*

(a) A is an IP* subset.

(b) A is a member of every idempotent of δS.

(c) $A \cap B$ is IP for every IP subset B of S.

Proof. (a) $\Rightarrow$ (b). Let p be an idempotent in δS. Suppose that $A \notin p$. Then $S \backslash A \in p$; and so $S \backslash A$ is an IP set that misses A. This is a contradiction.

(b) $\Rightarrow$ (c). Let B be an IP set in S and pick an idempotent $p \in \delta S$ such that $B \in p$. Then $A \in p$. Therefore $A \cap B \in p$. So $A \cap B$ is also IP.

(c) $\Rightarrow$ (a). Let B be an IP subset of S. Then $A \cap B$ is IP and therefore nonempty. So A is IP*. □

In the case of č-IP* sets a little more is required, and there is no immediate association with idempotents.

5.5 Theorem. *Let $(S,*)$ be an adequate partial semigroup and let $A \subseteq S$. The following statements are equivalent:*

(a) A is a č-IP* subset.

(b) $A \cap B$ is č-IP for every č-IP subset B of S.

Proof. (a) $\Rightarrow$ (b). Let B a č-IP subset of S. Pick $\langle x_n \rangle_{n=1}^{\infty}$ in S such that for all $F \in \mathcal{P}_f(\mathbb{N})$, $\prod_{n \in F} x_n$ is defined and $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq B$. Let

$$C_0 = \{F \in \mathcal{P}_f(\mathbb{N}) : \prod_{n \in F} x_n \in A\},$$

and let

$$C_1 = \{F \in \mathcal{P}_f(\mathbb{N}) : \prod_{n \in F} x_n \notin A\}.$$

So $\mathcal{P}_f(\mathbb{N}) = C_0 \cup C_1$. Then by [9, Corollary 5.17] there exists $i \in \{0,1\}$ and $\langle G_n \rangle_{n=1}^{\infty}$ with $\max G_n < \min G_{n+1}$ for all n, such that for all $H \in \mathcal{P}_f(\mathbb{N})$, $\bigcup_{n \in H} G_n \in C_i$. Let $y_n = \prod_{t \in G_n} x_t$. Then $FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq FP(\langle x_n \rangle_{n=1}^{\infty})$. If $i = 0$ then $FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq A$. If $i = 1$ then $FP(\langle y_n \rangle_{n=1}^{\infty}) \cap A = \emptyset$, in which case $FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq S \backslash A$. Then $S \backslash A$ is $\check{c}$-IP and since A is $\check{c}$-IP* this implies that $A \cap (S \backslash A) \neq \emptyset$. This is a contradiction. So $FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq A$ and $FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq B$. So $FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq A \cap B$. Thus $A \cap B$ is $\check{c}$-IP.

(b) $\Rightarrow$ (a). Suppose that A is not $\check{c}$-IP* and pick $B \subseteq S$ such that B is $\check{c}$-IP and $A \cap B = \emptyset$. We have already that $A \cap B$ is $\check{c}$-IP so pick $\langle x_n \rangle_{n=1}^{\infty}$ in S such that for all $F \in \mathcal{P}_f(\mathbb{N})$, $\prod_{n \in F} x_n$ is defined and $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A \cap B$. Then $A \cap B \neq \emptyset$. This is a contradiction. So A is $\check{c}$-IP*. □

The same pattern follows for PS* and $\check{c}$-PS*, where "PS" and "$\check{c}$-PS" abbreviate "piecewise syndetic" and "$\check{c}$-piecewise syndetic" respectively.

5.6 Theorem. *Let $(S, *)$ be an adequate partial semigroup and let $A \subseteq S$. The following statements are equivalent:*

(a) A is a PS subset.*

(b) $K(\delta S) \subseteq \overline{A}$.

(c) $A \cap B$ is piecewise syndetic for every piecewise syndetic subset B of S.

Proof. (a) $\Rightarrow$ (b) Suppose there exists $p \in K(\delta S)$ such that $A \notin p$. Pick such p. Then $S \backslash A \in p$ so $S \backslash A$ is a piecewise syndetic set which does not intersect A. This is a contradiction.

(b) $\Rightarrow$ (c) Let B be a piecewise syndetic subset of S. Then pick $q \in K(\delta S)$ such that $B \in q$. So $A \in q$. Therefore $A \cap B \in q$. Thus $A \cap B$ is piecewise syndetic.

(c) $\Rightarrow$ (a) Suppose that A is not PS*. Then by Theorem 5.1, $S \backslash A$ is a piecewise syndetic subset of S. But $A \cap (S \backslash A)$ is not piecewise syndetic. This is a contradiction. □

5.7 Theorem. *Let $(S,*)$ be an adequate partial semigroup and let $A \subseteq S$. The following statements are equivalent:*

(a) A is a č-PS subset of S.*

(b) For all $H \in \mathcal{P}_f(S)$ there exists $T \in \mathcal{P}_f(S)$ such that for all $x \in \sigma(T)$, there exists $y \in T \cap \sigma(H)$ such that $H * y * x \subseteq A$.

(c) For all $p \in K(\delta S)$, $\beta S * p \subseteq A$.

(d) $A \cap B$ is č-piecewise syndetic for every č-piecewise syndetic subset B of S.

Proof. (a) ⇒ (b) Given $H \in \mathcal{P}_f(S)$, given $T \in \mathcal{P}_f(S)$, and given $x \in \sigma(T)$, the assertion that $\big(T\cap\sigma(H)\big)*x \setminus \bigcup_{t\in H} t^{-1}(S\setminus A) \neq \emptyset$ is exactly the assertion that there exists $y \in T \cap \sigma(H)$ such that $H * y * x \subseteq A$. Thus statement (b) is exactly the assertion that $S\setminus A$ is not č-piecewise syndetic. Therfore (a) and (b) are equivalent by Theorem 5.1.

(a) ⇒ (c) Let $p \in K(\delta S)$ and assume that $(\beta S * p)\setminus\overline{A} \neq \emptyset$. Then $(\beta S * p) \cap \overline{(S\setminus A)} \neq \emptyset$. So $S\setminus A$ is č-piecewise syndetic by Theorem 3.8, a contradiction.

(c) ⇒ (d) Let B be a č-piecewise syndetic subset of S and pick by Theorem 3.8 some $p \in K(\delta S)$ such that $(\beta S{*}p) \neq \emptyset$. Then $(\beta S{*}p) \subseteq \overline{A}$, so $(\beta S{*}p)\cap\overline{(A \cap B)} \neq \emptyset$. Thus $A \cap B$ is piecewise syndetic.

(d) ⇒ (a) Trivial. □

We now turn our attention to central* sets and their relationship to IP* and č-IP* sets.

In a semigroup the natural relationship between IP and central sets and their characterizations in terms of idempotents, is transferred to the dual notions IP* and central*. These relationships are discussed in detail in [9]. We provide the basic properties here for easy referencing.

5.8 Theorem. *Let S be a semigroup and let $A \subseteq S$. Then the following statements are equivalent:*

(a) A is a central subset of S.*

(b) A is a member of every minimal idempotent of βS.

(c) $A \cap B$ is a central set for every central subset B of S.

Proof. [9, Lemma 15.4].

Recall that the notion of central in an adequate partial is unilateral, and therefore we refer only to central* sets here.

5.9 Theorem. *Let $(S, *)$ be an adequate partial semigroup and let $A \subseteq S$. Then the following statements are equivalent:*

(a) A is a central subset of S.*

(b) A is a member of every minimal idempotent of δS.

(c) $A \cap B$ is a central set for every central subset B of S.

Proof. (a) $\Rightarrow$ (b). Let p be a minimal idempotent in δS. If $A \notin p$ then $S \backslash A \in p$. So $S \backslash A$ is a central set that misses A. This is a contradiction.

(b) $\Rightarrow$ (c). Let C be a central subset of S. Pick a minimal idempotent $p \in \delta S$ such that $C \in p$. By assumption $A \in p$, so $A \cap C \in p$. Therefore $A \cap C$ is central.

(c) $\Rightarrow$ (a). Let C be a central subset of S. Then $A \cap C$ is central. So pick a minimal idempotent $p \in \delta S$ such that $A \cap C \in p$. Then $A \cap C \neq \emptyset$. Thus A is central*. $\square$

An immediate consequence of Theorems 5.4, 5.5, 5.6, 5.7, and 5.9 is the following:

5.10 Theorem. *Let $(S, *)$ be an adequate partial semigroup and let A and B be subsets of S.*

(a) If A and B are IP sets, then $A \cap B$ is an IP* set,*

(b) If A and B are č-IP sets, then $A \cap B$ is a č-IP* set,*

(c) If A and B are PS sets, then $A \cap B$ is a PS* set,*

(d) If A and B are č-PS sets, then $A \cap B$ is a č-PS* set,*

(e) If A and B are central sets, then $A \cap B$ is a central* set,*

(f) If A is an IP set and B is a č-IP* set, then $A \cap B$ is an IP* set.*

(g) If A is a PS set and B is a č-PS* set, then $A \cap B$ is a PS* set.*

Proof. (a) Let C be an IP set in S. Then by Theorem 5.4 $B \cap C$ is an IP set. So $A \cap (B \cap C) \neq \emptyset$. That is, $(A \cap B) \cap C \neq \emptyset$. So $A \cap B$ is IP*.

(b) Let C be a č-IP set in S. Then by Theorem 5.5, $B \cap C$ is a č-IP set. So $A \cap (B \cap C) \neq \emptyset$. That is, $(A \cap B) \cap C \neq \emptyset$. So $A \cap B$ is č-IP*.

(c) Let C be a piecewise syndetic set in S. Then by Theorem 5.6 $B \cap C$ is a piecewise syndetic set. So $A \cap (B \cap C) \neq \emptyset$. That is, $(A \cap B) \cap C \neq \emptyset$. So $A \cap B$ is PS*.

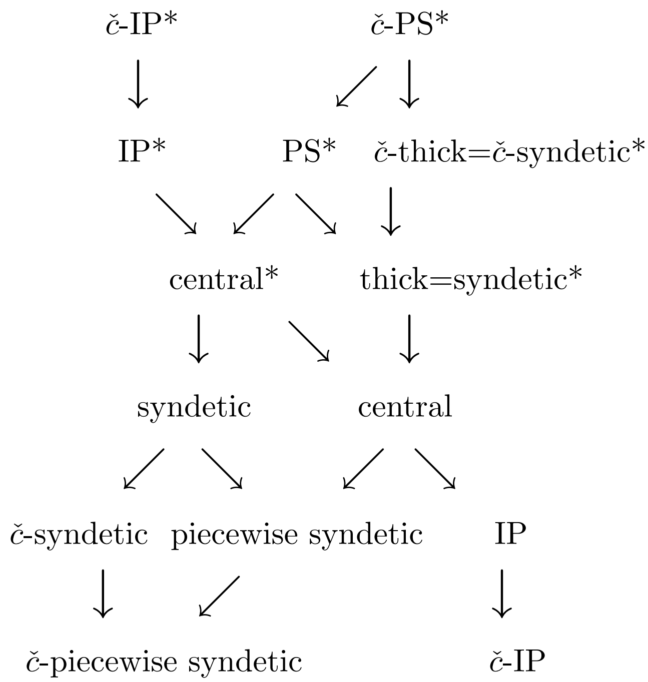
(d) Let C be a č-piecewise syndetic set in S. Then by Theorem 5.7, $B \cap C$ is a č-piecewise syndetic set. So $A \cap (B \cap C) \neq \emptyset$. That is, $(A \cap B) \cap C \neq \emptyset$. So $A \cap B$ is č-PS*.

(e) Let C be a central set in S. Then by Theorem 5.9, $B \cap C$ is a central set. So, again by Theorem 5.9 $A \cap (B \cap C) \neq \emptyset$. That is, $(A \cap B) \cap C \neq \emptyset$. So $A \cap B$ is central*.

(f) By Theorems 4.6 and 5.2, this reduces to part (a) above.

(g) By Theorems 3.10 and 5.2, this reduces to part (c) above. □

We will now consider the pattern of implications that exist for the notions of size we have presented, in an adequate partial semigroup. We conclude by showing that none of the missing implications are valid. The examples in the semigroup $(\mathbb{N}, +)$ are taken from [2], where they were presented without proof. These examples use the fact that every semigroup is a partial semigroup.



5.11 Theorem. *Let $(S,*)$ be an adequate partial semigroup and $A \subseteq S$. Then all of the implications indicated in the above diagram hold.*

Proof. Starting from the top left corner of the diagram, we have that $\check{c}$-IP* implies IP* by Theorems 4.6 and 5.2.

Given that A is $\check{c}$-PS*, by Theorems 3.10 and 5.2, A is PS*.

If A is $\check{c}$-PS*, then by Corollary 3.9 and Theorem 5.2, A is $\check{c}$-thick.

Given that A is IP*, we have by Theorem 5.3 that A is an element of every idempotent of δS. In particular A is an element of every minimal idempotent and therefore A is central* by Theorem 5.9.

If A is PS*, then by Theorem 5.6, $K(\delta S) \subseteq A$ so $E\big(K(\delta S)\big) \subseteq A$. Thus A is central* by Theorem 5.9. Also A is syndetic* by Lemma 3.4 and Theorem 5.2.

If A is $\check{c}$-thick, then by Theorem 2.17 A is thick.

To see that central* implies syndetic, let A be central* and let $p \in \delta S$. By Theorem 1.20, pick a minimal left ideal $M \subseteq \delta S * p$ and pick a minimal idempotent $e \in M \subseteq \delta S * p$. We have that $e \in \overline{A}$ by Theorem 5.9, so $\overline{A} \cap \delta S * p \neq \emptyset$. That is,

A is syndetic.

Given a central* set A of S, by Theorem 5.9, A is contained in every minimal idempotent of δS. Therefore A is contained in some minimal idempotent, and therefore A is central.

If A is a thick set then $\delta S * p \subseteq \overline{A}$ for some $p \in \delta S$. Pick by Theorem 1.20 a minimal left ideal $L \subseteq \delta S * p$ and an idempotent $e \in \delta S$ such that $e \in L$. That is, $e \in L \subseteq \delta S * p \subseteq \overline{A}$. So $A \in e$. Therefore A is central.

Syndetic implies č-syndetic by Theorem 2.27.

Syndetic implies piecewise syndetic by Lemma 3.4.

Given a central set A, $\overline{A}$ contains a minimal idempotent of δS. Therefore $\overline{A} \cap K(\delta S) \neq \emptyset$. Thus A is piecewise syndetic. Also since A is a member of a minimal idempotent, A is a member of some idempotent of δS so A is IP.

We have that č-syndetic implies č-piecewise syndetic by Corollary 3.9.

Piecewise syndetic implies č-piecewise syndetic by Theorem 3.10.

We have that IP implies č-IP by Theorem 4.6. □

We conclude this dissertation by showing that none of the missing implications is valid in general. In fact we establish a stronger result. We show in the following theorems that for each of the listed properties there exist an adequate partial semigroup and a subset A of S which has that property and only those other properties that are forced by the listed implications.

5.12 Theorem. *There exist an adequate partial semigroup $(S, *)$ and a subset A of S such that A is č-IP but is neither IP nor č-piecewise syndetic.*

Proof. Let S and $\mathcal{A}$ be as in the proof of Theorem 4.5. So $S = \{A \subseteq \mathbb{N} : A \neq \emptyset \text{ and } |A \backslash 2\mathbb{N}| < \omega\}$ and $\mathcal{A} = \mathcal{P}_f(2\mathbb{N})$. Define $\uplus$ on S such that

$$A \uplus B = \begin{cases} A \cup B & \text{if } A \cap B = \emptyset \\ \text{undefined} & \text{if } A \cap B \neq \emptyset \end{cases}$$

We saw in Theorem 4.5 that $\mathcal{A}$ is $\check{c}$-IP and not IP. We claim that $\mathcal{A}$ is not $\check{c}$-piecewise syndetic. Let $\mathcal{H} \in \mathcal{P}_f(S)$. Then $\bigcup \mathcal{H}$ has only finitely many odd members. Pick $y \in \big(2\mathbb{N}+1\big) \setminus (\bigcup \mathcal{H})$. Then $\{y\} \in \sigma(\mathcal{H})$. Let $\mathcal{T} = \big\{\{y\}\big\}$. Pick $X \in \sigma(\mathcal{T})$. Then $\{y\} \in \mathcal{T} \cap \sigma(\mathcal{H})$. We claim that $\{y\} * X \notin \bigcup_{T \in \mathcal{H}} T^{-1}\mathcal{A}$. To see this, let $T \in \mathcal{H}$ and suppose that $T * \{y\} * X \in \mathcal{A}$. But $y \in T * \{y\} * X$, so $T * \{y\} * X \notin \mathcal{A}$. Therefore $\mathcal{A}$ is not $\check{c}$-piecewise syndetic. □

5.13 Theorem. *There exist a partial semigroup S and a subset A of S which is $\check{c}$-piecewise syndetic but none of $\check{c}$-syndetic, piecewise syndetic, or $\check{c}$-IP.*

Proof. Let $S = FP(\langle y_t \rangle_{t=1}^{\infty})$ as in Theorem 3.7, where $\prod_{t \in F} y_t * \prod_{t \in G} y_t$ is defined exactly when $\max F < \min G$. Let

$$A = \{\textstyle\prod_{t \in F} y_t :\ 1 \in F, |F| \geq 2,\ \text{and}\ \max F > \max(F \backslash \{\max F\}) + 2\}\,.$$

To see that A is $\check{c}$-piecewise syndetic, let $H = \{y_1\}$. Let $T \in \mathcal{P}_f(S)$ be given. If $T \cap \sigma(H) = \emptyset$ we are done. Assume that $T \cap \sigma(H) \neq \emptyset$. For each $z \in T \cap \sigma(H)$, pick $F_z \in \mathcal{P}_f(\mathbb{N} \backslash \{1\})$ such that $z = \prod_{t \in F_z} y_t$. Pick $m > \max\big(\bigcup_{z \in T \cap \sigma(H)} F_z\big) + 2$. then $y_1 * (T \cap \sigma(H)) * y_m \subseteq A$. So A is $\check{c}$-piecewise syndetic.

Since each member of A begins with y_1 their products in S are not defined. Thus A is not $\check{c}$-IP.

Notice that A is a subset of the example in the proof of Theorem 3.7, so A is not piecewise syndetic.

To see that A is not $\check{c}$-syndetic, let $H \in \mathcal{P}_f(S)$ be given. For each $z \in H$ pick $F_z \in \mathcal{P}_f(\mathbb{N})$ such that $z = \prod_{t \in F_z} y_t$. Pick $m \in \mathbb{N}$ such that $m > \max\big(\bigcup_{z \in H} F_z\big)$. Let $x = y_m * y_{m+1}$. Then $x \in \sigma(H)$ and for all $z \in H,\ z * x \notin A$. Therefore A is not $\check{c}$-syndetic. □

5.14 Theorem. *There exist an adequate partial semigroup $(S, *)$ and a subset A of S such that A is an IP set but A is not $\check{c}$-piecewise syndetic.*

Proof. Let $(S,*)=(\mathbb{N},+)$ and let $A=FS(\langle 2^{2n}\rangle_{n=1}^{\infty})$ Then A is an IP-set. To see that A is not $\check{c}$-piecewise syndetic, let $H\in\mathcal{P}_f(S)$ and pick $n\in\mathbb{N}$ such that $\max H<2^{2n}$. Let $F=\{1,2,\ldots,2^{2n+2}\}$. Let x be given and find $s\in\{1,2,\ldots,2^{2n+2}\}$ such that $x+s\equiv 2^{2n+1}(\text{mod }2^{2n+2})$, then $x+s=a\cdot 2^{2n+2}+2^{2n+1}$ for some $a\in\omega$. For $t\in\{1,2,\ldots,2^{2n}\}$, $x+s+t\notin A$. Therefore A is not $\check{c}$-piecewise syndetic. □

5.15 Theorem. *There exist an adequate partial semigroup $(S,*)$ and a subset A of S such that A is piecewise syndetic but is neither $\check{c}$-syndetic nor $\check{c}$-IP.*

Proof. Let $(S,*)=(\mathbb{N},+)$ and $A=\{2^n+2m-1: n,\ m\in\mathbb{N}\text{ and }m<n\}$. To see that A is piecewise syndetic, which is the same as $\check{c}$-piecewise syndetic in $(\mathbb{N},+)$, let $H=\{1,2\}$, let $F\in\mathcal{P}_f(\mathbb{N})$, and let $n=\max F$. Then $F+2^n\subseteq\bigcup_{t\in H}(-t+A)$. Therefore A is piecewise syndetic.

To see that A is not $\check{c}$-syndetic let $H\in\mathcal{P}_f(\mathbb{N})$ be given. Pick $m\in\mathbb{N}$ such that $2^m>\max H$. Let $x=2^{m+1}+2m$. Then for $t\in H$, $2^{m+1}+2m<x+t<2^{m+1}+2^m+2m<2^{m+2}+1$. So $x+t\notin A$. Therefore A is not $\check{c}$-syndetic.

To see that A is not $\check{c}$-IP notice that the sum of any two members of A is not in A. □

5.16 Theorem. *There exist an adequate partial semigroup $(S,*)$ and a subset A of S such that A is $\check{c}$-syndetic is neither piecewise syndetic nor $\check{c}$-IP.*

Proof. Let S and A be as in the proof of Theorem 2.21. So let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in a semigroup $(S,\cdot)$ which satisfies uniqueness of finite products. Let $S=FP(\langle x_n\rangle_{n=1}^{\infty})=\{\prod_{n\in F}x_n: F\in\mathcal{P}_f(\mathbb{N})\}$ and define

$$(\textstyle\prod_{n\in F}x_n)*(\prod_{n\in G}x_n) \quad = \quad \begin{cases}\prod_{n\in F\cup G}x_n & \text{if } \max F<\min G\\ \text{undefined} & \text{if } \max F\geq\min G\,.\end{cases}$$

And let $A=\{\prod_{n\in F}x_n: F\in\mathcal{P}_f(\mathbb{N})\text{ and }1\in F\}$. We have already seen that A is $\check{c}$-syndetic.

We also saw in Theorem 2.21 that for every $p\in\delta S$, $\overline{A}\cap(\delta S*p)=\emptyset$. Suppose that there exists $q\in\delta S$ such that $q\in K(\delta S)\cap\overline{A}$. Pick a minimal left ideal L of S

such that $q \in L$. Also pick an idempotent $e \in L$. Then $q = q * e$ so $q \in (\delta S * e) \cap \overline{A}$. This is a contradiction. Therefore A is not piecewise syndetic.

Since $y * z$ is undefined for all $y, z \in A$, A is not č-IP. □

5.17 Theorem. *There exist an adequate partial semigroup $(S, *)$ and a subset A of S such that A is central but is neither č-syndetic nor thick.*

Proof. Let $(S, *) = (\mathbb{N}, +)$ and $A = \{2^n + 2m : n,\ m \in \mathbb{N} \text{ and } m < n\}$. To see that A is central notice that $A = B \cap C$, where $B = \{2^n + m : n,\ m \in \mathbb{N} \text{ and } m < 2n\}$, and $C = \{2m :\ m \in \mathbb{N}\}$. Note that B is thick and C is IP*. So pick a minimal left ideal L of δS such that $L \subseteq \overline{B}$. Pick $p = p + p \in L$. Since C is IP*, $C \in p$. Then $p \in \overline{B \cap C} = \overline{A}$. Therefore A is central.

Since $A \subseteq 2\mathbb{N}$, A is not thick.

To see that A is not č-syndetic let $H \in \mathcal{P}_f(\mathbb{N})$ be given. Pick $n \in \mathbb{N}$ such that $2^n > \max H$. Let $x = 2^{n+1} + 2n$. Then for $t \in H$, $2^{2n+1} + 2n < x + t < 2^{n+1} + 2^n + 2n < 2^{n+2} + 1$. So $x + t \notin A$. Therefore A is not č-syndetic. □

5.18 Theorem. *There exist an adequate partial semigroup $(S, *)$ and a subset A of S such that A is syndetic but not č-IP.*

Proof. Let $(S, *) = (\mathbb{N}, +)$ and $A = 2\mathbb{N} + 1$. It is clear that A is syndetic since there are no gaps of length longer than 1.

To see that A is not č-IP notice that the sum of any two members of A is a member of $2\mathbb{N}$. □

5.19 Theorem. *There exist a partial semigroup S and a subset A of S such that A is thick but neither č-syndetic nor č-thick.*

Proof. Let $S = FP(\langle y_t \rangle_{t=1}^{\infty})$ with the products defined as in Theorem 2.14. Let $A = \{\prod_{t \in F} y_t :\ |F| \geq 2,\ 1 \notin F,\ \max F \text{ is even and } \max F > \max(F \backslash \{\max F\}) + 2\}$.

To see that A is thick, let $p \in \delta S \cap c\ell\{y_{2n} : n \in \mathbb{N}\}$. We claim that $\delta S * p \subseteq A$. For this it suffices to show that $\varphi(y_1) \subseteq \{x \in S : x^{-1}A \in p\}$. To see this, let $x \in \varphi(y_1)$. Then $x = \prod_{n \in H} y_n$ for some $H \in \mathcal{P}_f(\mathbb{N}\backslash\{1\})$. Let $l = \max H$. Then $\{y_{2n} : n > l+2\} \in p$, and $\{y_{2n} : n > l+2\} \subseteq x^{-1}A$. Therefore A is thick.

For no $x \in S$ is $\{y_1\} * x \in A$. Thus A is not č-thick.

To see A is not č-syndetic, let $H \in \mathcal{P}_f(S)$. Pick $m \in \mathbb{N}$ such that $H \subseteq FP(\langle y_n \rangle_{n=1}^{m})$. Let $z = y_{m+1} * y_{m+2}$. Then $z \in \sigma(H)$ and $z \notin \bigcup_{t \in H} t^{-1}A$. Therefore A is not č-syndetic. □

5.20 Theorem. *There exist an adequate partial semigroup S and a subset A of S such that A is central* but neither IP* nor thick.*

Proof. Let $(S,*) = (\mathbb{N},+)$ and $A = 2\mathbb{N}\backslash FS(\langle 2^{2n} \rangle_{n=1}^{\infty})$. To see that A is central*, recall that we saw in Theorem 5.14 that $FS(\langle 2^{2n} \rangle_{n=1}^{\infty})$ is not piecewise syndetic, and thus $K(\delta S) \subseteq \overline{\mathbb{N}\backslash FS(\langle 2^{2n} \rangle_{n=1}^{\infty})}$. Since also $2\mathbb{N}$ is IP*, $\overline{A}$ contains all the minimal idempotents of δS.

The set A is not IP* since $FS(\langle 2^{2n} \rangle_{n=1}^{\infty})$ is an IP set that misses A.

Since $A \subseteq 2\mathbb{N}$, A is not thick. □

5.21 Theorem. *There exist an adequate partial semigroup S and a subset A of S such that A is č-thick but not č-syndetic.*

Proof. Let $(S,*) = (\mathbb{N},+)$ and $A = \{2^n + m : n,\ m \in \mathbb{N} \text{ and } m < n\}$. Clearly A is thick. One shows that A is not č-syndetic as in the proof of Theorem 5.17. □

5.22 Theorem. *There exist an adequate partial semigroup S and a subset A of S such that A is PS* but not č-thick nor IP*.*

Proof. Let $S = FP(\langle y_t \rangle_{t=1}^{\infty})$ with the products defined as in Theorem 2.14. Let $A = FP(\langle y_n \rangle_{n=2}^{\infty})\backslash FP(\langle y_{2n} \rangle_{n=1}^{\infty})$. By Lemma 4.7, $\bigcap_{m=1}^{\infty} \overline{FP(\langle y_{2n} \rangle_{n=m}^{\infty})} \cap \delta S$ is a subsemigroup of δS. So pick $p \in \bigcap_{m=1}^{\infty} \overline{FP(\langle y_{2n}\rangle_{n=m}^{\infty})}$. Then $FP(\langle y_{2n}\rangle_{n=1}^{\infty}) \in p$. So A is not IP*.

To see A is not č-thick, we show that $S\backslash A$ is č-syndetic. Let $H = \{y_1\}$. Then for all $x \in \sigma(H)$, $y_1 * x \in S\backslash A$. Therefore $\sigma(H) \subseteq \bigcup_{t\in H} t^{-1}(S\backslash A)$. So $S\backslash A$ is č-syndetic. Then by Theorem 2.24 A is not č-thick.

Now let $p \in K(\delta S)$. We show that $A \in p$. Notice that $FP(\langle y_{2n}\rangle_{n=2}^{\infty}) = \varphi(y_1)$. So $FP(\langle y_{2n}\rangle_{n=2}^{\infty}) \in p$. To show $FP(\langle y_{2n}\rangle_{n=1}^{\infty}) \notin p$ we show that $FP(\langle y_{2n}\rangle_{n=1}^{\infty})$ is not piecewise syndetic; for which it suffices to show that $B = FP(\langle y_{2n}\rangle_{n=1}^{\infty})$ is not č-piecewise syndetic, by Theorem 3.10. Suppose there exists $H \in \mathcal{P}_f(S)$ such that for all $T \in \mathcal{P}_f(S)$ there exists $x \in \sigma(T)$ such that $\big(T \cap \sigma(H)\big) * x \subseteq \bigcup_{t\in H} t^{-1}A$. Pick $m \in \mathbb{N}$ such that $H \subseteq FP(\langle y_t\rangle_{t=1}^{m})$. Let $T = \{y_{2m+1}\}$. Then $T \subseteq \sigma(H)$. So for all $x \in \sigma(T)$ and for all $t \in H$, $t * y_{2m+1} * x \notin B$. This is a contradiction. So $A \in p$ and therefore A is PS*. □

5.23 Theorem. *There exist an adequate partial semigroup S and a subset A of S such that A is IP* but not thick nor č-IP*.*

Proof. Let $S = \{B \subseteq \mathbb{N} :\ B \neq \emptyset \text{ and } |B\backslash 2\mathbb{N}| < \omega\}$, as in Theorem 4.5. Further, let $\mathcal{A} = S\backslash(\mathcal{P}_f(2\mathbb{N}) \cup \{B \subseteq \mathbb{N} :\ |B\backslash 2\mathbb{N}| \in 2\mathbb{N}-1\})$. Notice that $FP(\langle\{2n\}\rangle_{n=1}^{\infty}) = \mathcal{P}_f(2\mathbb{N})$. So $\mathcal{A}$ is not č-IP*.

Let $p \in \delta S$ such that $p \circledast p = p$. We show that $\mathcal{A} \in p$. We have that $\varphi(2\mathbb{N}) \in p$ and $\varphi(2\mathbb{N}) \cap \mathcal{P}_f(2\mathbb{N}) = \emptyset$. So $\mathcal{P}_f(2\mathbb{N}) \notin p$. Suppose that $\mathcal{D} = \{B \subseteq \mathbb{N} :\ |B\backslash 2\mathbb{N}| \in 2\mathbb{N}-1\} \in p$. Pick $B \in \mathcal{D}$ such that $B^{-1}\mathcal{D} \in p$. Pick $C \in \mathcal{D} \cap B^{-1}\mathcal{D}$. Then B, C, and $B \circledast C$ each have an odd number of odd integers. This is a contradiction. So $\mathcal{A}$ is IP*.

To see that $\mathcal{A}$ is not thick, suppose that for some $p \in \delta S$, $\delta S * p \subseteq \overline{\mathcal{A}}$. Let $\mathcal{D} = \{B \subseteq \mathbb{N} :\ |B\backslash 2\mathbb{N}| \in 2\mathbb{N}-1\}$.

Case 1: $\mathcal{D} \in p$. It is easy to see that we can pick $q \in \delta S$ such that $S\backslash\mathcal{D} \in q$. Then $\mathcal{D} \in q * p$. Thus, $\mathcal{A} \notin q * p$.

Case 2: $\mathcal{D} \notin p$. Pick $q \in \delta S$ such that $\mathcal{D} \in q$. Then $\mathcal{D} \in q * p$. □

5.24 Theorem. *There exist an adequate partial semigroup S and a subset A of S such that A is č-PS* but not IP*.*

Proof. Let $(S, *) = (\mathbb{N}, +)$ and $A = \mathbb{N} \backslash FS(\langle 2^{2n} \rangle_{n=1}^{\infty})$. The set A is $\check{c}$-PS* since, as we saw in Theorem 5.14, $FS(\langle 2^{2n} \rangle_{n=1}^{\infty})$ is not $\check{c}$-piecewise syndetic. Therefore A misses no $\check{c}$-piecewise syndetic set in S.

Since A misses the IP set $FS(\langle 2^{2n} \rangle_{n=1}^{\infty})$, A is not IP*. □

5.25 Theorem. *There exist an adequate partial semigroup S and a subset A of S such that A is č-IP* but not thick.*

Proof. Let $(S, *) = (\mathbb{N}, +)$ and $A = 2\mathbb{N}$. □

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