Hyperbananas
A Family of Flexible Frameworks

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\(^1\)Supported by NSF grant DMS-0849637
A bar and joint framework is a simple graph \( G = (V, E) \) with an embedding function \( p : V \rightarrow \mathbb{R}^d \).

Click to start
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- The embedding determines the position of joints
- How do we determine if a framework is rigid or flexible?
Bar-and-Joint Rigidity

- Examine **internal motions** and **rigid motions**

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- Rigid motions are distance preserving (translations, rotations)
- Internal motions change the distance between at least one pair of vertices
- **Rigid** frameworks admit only rigid motions
- **Flexible** frameworks admit internal motions as well
Degrees of Freedom

- The number of “basic” internal motions

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- Alternatively, number of bars to be rigid
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- The number of “basic” internal motions
- Alternatively, number of bars to be rigid
- Triangle in $\mathbb{R}^2$ has 0 degrees of freedom $\Rightarrow$ it is rigid
- Quadrilateral in $\mathbb{R}^2$ has 1 degree of freedom $\Rightarrow$ it is flexible
Maxwell Conditions

- Rigidity gives us a combinatoric constraint
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**Definition**

A graph $G = (V, E)$ embedded in $\mathbb{R}^2$ is a **Maxwell graph** if it satisfies the following conditions.

1. $|E| = 2|V| - 3$
2. $|E(V')| \leq 2|V'| - 3$, for all $V' \subseteq V$ where $|V'| \geq 2$
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Definition

A graph $G = (V, E)$ embedded in $\mathbb{R}^d$ is a Maxwell graph if it satisfies the following conditions.

1. $|E| = d|V| - \binom{d+1}{2}$
2. $|E(V')| \leq d|V'| - \binom{d+1}{2}$, for all $V' \subseteq V$ where $|V'| \geq d$

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Double Banana

- Is it true that G is a Maxwell graph $\Rightarrow G$ is rigid as well?
Double Banana

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- Satisfies the condition $|E| = 3|V| - 6$
- However, there is a hinge between two black vertices
- We call this an implied edge
Research Questions

- Open Question: Find a necessary and sufficient combinatorial condition for rigidity in $\mathbb{R}^3$. 
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What other classes of counterexamples are there?

My question: can we generalize current counterexamples to $d$-dimensional space? Specifically, can we extend the double banana example?

Goal: try to use implied edges to connect rigid components.
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Goal: try to use implied edges to connect rigid components
Can we make a generalized double banana in $\mathbb{R}^4$?
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Problem: need to add two edges to make it Maxwell

Want to find an example that doesn’t need these edges
Can we make a generalized double banana in $\mathbb{R}^5$?
5D Banana

- Can we make a generalized double banana in $\mathbb{R}^5$?
- Yes, and without any extra edges between complete graphs!
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Can we generalize this?
Hyperbananas

Lives in $d$-dimensional space for odd $d$
Hyperbananas

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Made up of vertices from two $K_d$ graphs, and $n$ banana vertices
Hyperbananas

- Lives in $d$-dimensional space for odd $d$
- Made up of vertices from two $K_d$ graphs, and $n$ banana vertices
- Each banana vertex connects to all vertices except other banana vertices

$KB_{d,n}$
Hyperbananas

- Lives in $d$-dimensional space for odd $d$
- Made up of vertices from two $K_d$ graphs, and $n$ banana vertices
- Each banana vertex connects to all vertices except other banana vertices
- It must be that $n = \frac{d+1}{2}$
$KB_{3,2}$ is just the classical double banana example.
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$KB_{5,3}$

- We saw $KB_{5,3}$ as the five dimensional example earlier
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Results

**Theorem**

The hyperbanana $KB_{d,n}$ embedded in $\mathbb{R}^d$ where $n = \frac{d+1}{2}$ is a flexible Maxwell graph with $\binom{n}{2}$ degrees of freedom.
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- Degrees of freedom coincide with the $ K_n $ of implied edges.
Theorem

The hyperbanana \( KB_{d,n} \) embedded in \( \mathbb{R}^d \) where \( n = \frac{d+1}{2} \) is a flexible Maxwell graph with \( \binom{n}{2} \) degrees of freedom.

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Sketch of Proof

- Maxwell: combinatorial argument
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- Flexibility: Involves analysis of the **rigidity matrix**
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- The upper/lower halves of the matrix each correspond to a rigid component of $KB_{d,n}$
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\[
\begin{bmatrix}
V_2 & V_1 & V_3 \\
M'(1) & 0 \\
0 & M'(2)
\end{bmatrix}
\]
Sketch of Proof

- Maxwell: combinatorial argument
- Flexibility: Involves analysis of the **rigidity matrix**
- The upper/lower halves of the matrix each correspond to a rigid component of $KB_{d,n}$

\[
\begin{bmatrix}
  & V_2 & V_1 & V_3 \\
M'(1) & 0 & \\
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\]

- Row reduction reveals dependencies in the matrix that correspond to the implied edges
The 4D banana example can be generalized.
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Like $KB_{d,n}$, but with $\frac{d}{2}$ edges carefully added
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- Proven they are Maxwell, conjectured that they are flexible
Future Work

- The 4D banana example can be generalized

- Like $KB_{d,n}$, but with $\frac{d}{2}$ edges carefully added
- Proven they are Maxwell, conjectured that they are flexible
- Symmetry is lost when $\frac{d}{2}$ edges are added, so rigidity matrix analysis is more delicate
Thanks for listening!

Questions?

Thanks to my advisors Audrey Lee-St. John and Jessica Sidman for their guidance on my research.

Also, thanks to the Michigan REU and Patrick Boland for hosting/advising the REU program which I participated in last summer.