Hyperbananas

A Family of Flexible Frameworks

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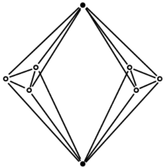
University of Michigan
Mount Holyoke REU



[1]Supported by NSF grant DMS-0849637

Bar-and-Joint Rigidity

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- The embedding determines the position of joints
- How do we determine if a framework is rigid or flexible?

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- **Rigid** frameworks admit only rigid motions
- **Flexible** frameworks admit internal motions as well

Degrees of Freedom

- The number of “basic” internal motions

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- Triangle in $\mathbb{R}^2$ has 0 degrees of freedom $\Rightarrow$ it is rigid
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Maxwell Conditions

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A graph $G = (V, E)$ embedded in $\mathbb{R}^2$ is a **Maxwell graph** if it satisfies the following conditions.

1. $|E| = 2|V| - 3$
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A graph $G = (V, E)$ embedded in $\mathbb{R}^d$ is a **Maxwell graph** if it satisfies the following conditions.

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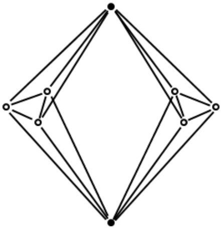

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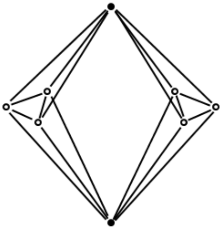

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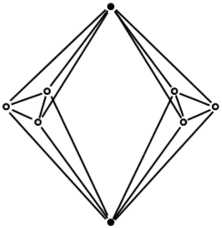

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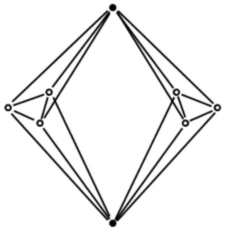

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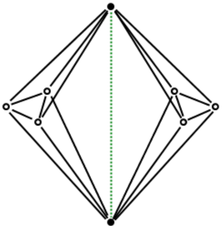

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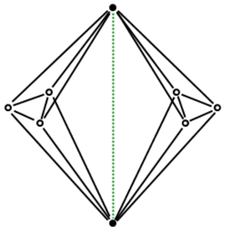

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- We call this an **implied edge**

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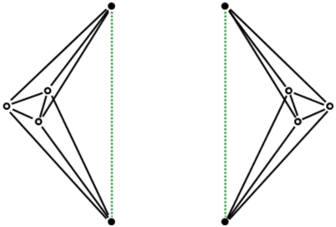

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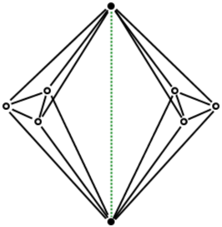

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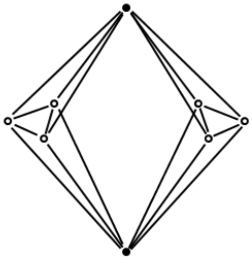

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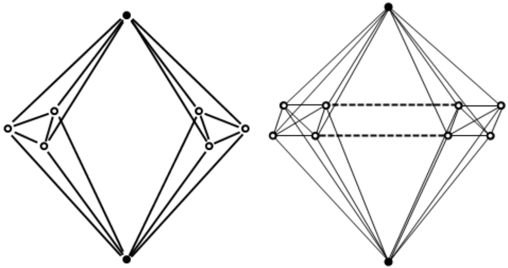

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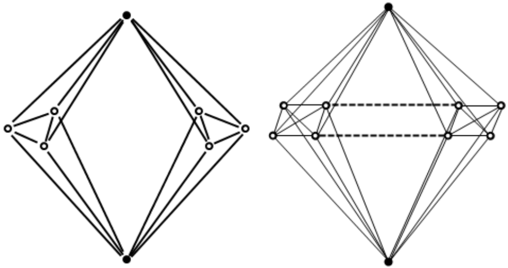

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Combinatorial Rigidity

Motivation for Research

Hyperbananas

Results and Future Work

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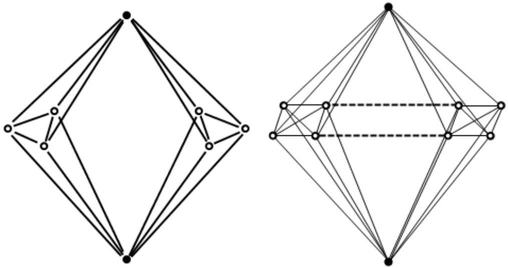

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- Want to find an example that doesn't need these edges

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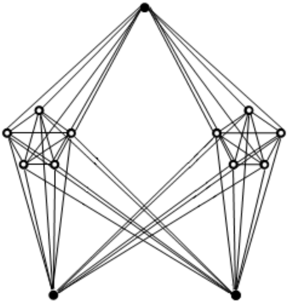

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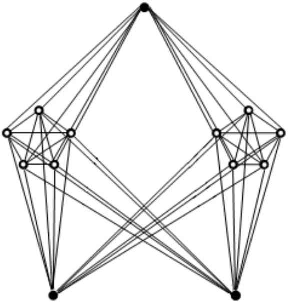

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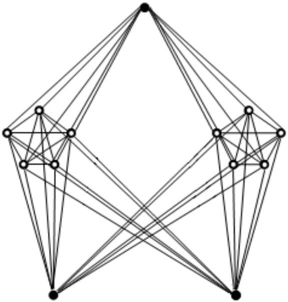

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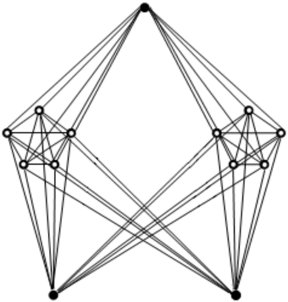

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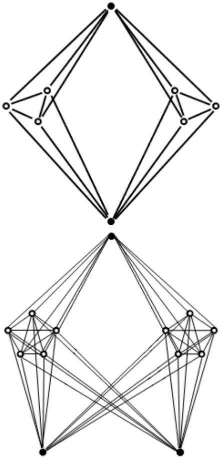
- Can we generalize this?

Hyperbananas



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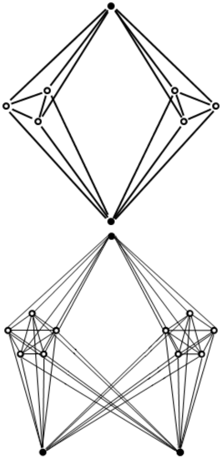

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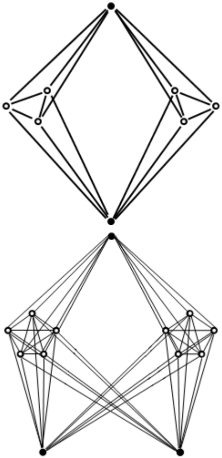

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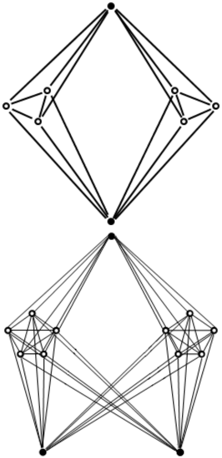

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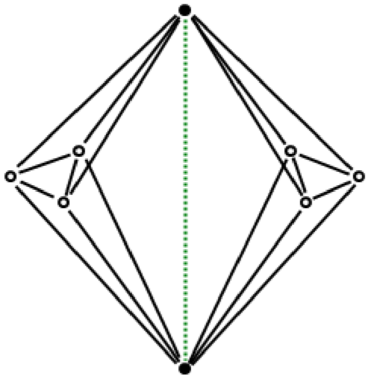

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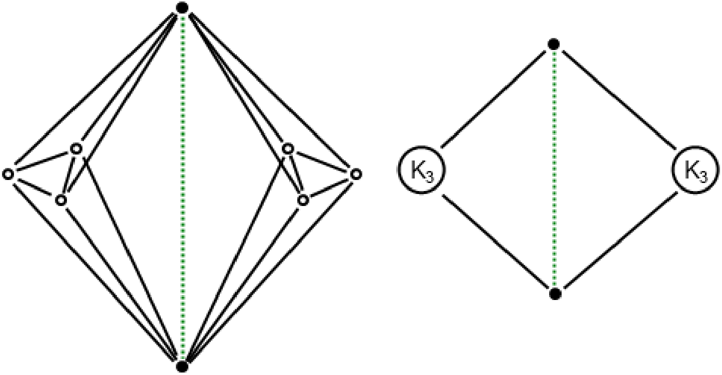

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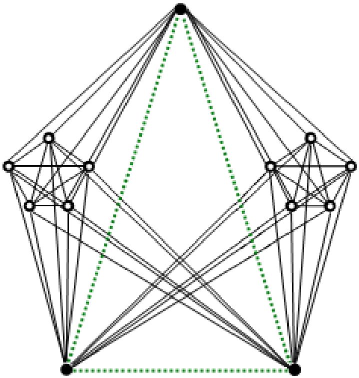

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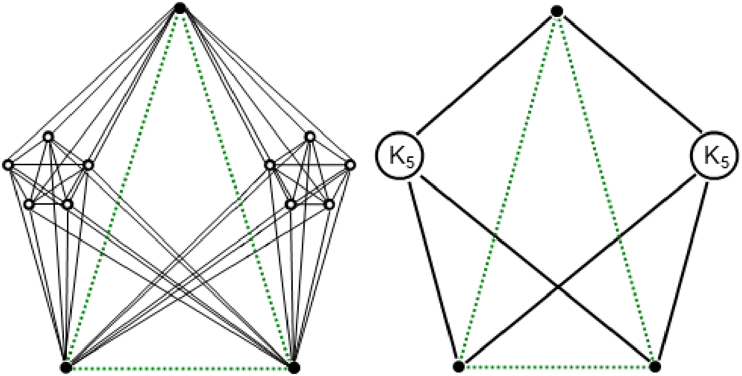

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The hyperbanana $KB_{d,n}$ embedded in $\mathbb{R}^d$ where $n = \frac{d+1}{2}$ is a flexible Maxwell graph with $\binom{n}{2}$ degrees of freedom.

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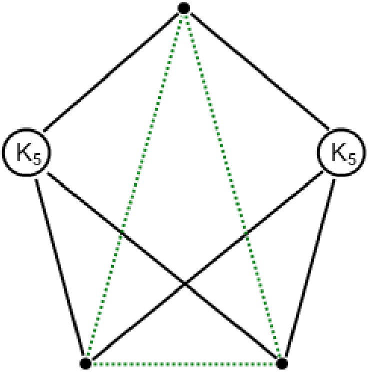

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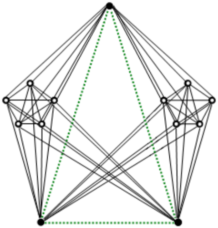

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- Row reduction reveals dependencies in the matrix that correspond to the implied edges

Future Work

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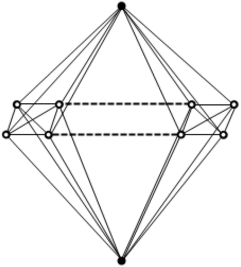

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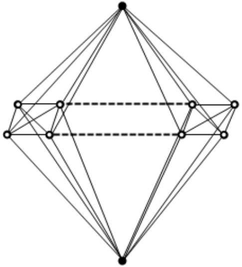

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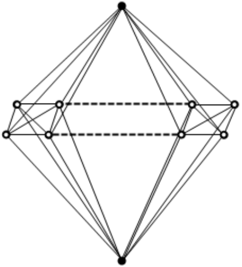

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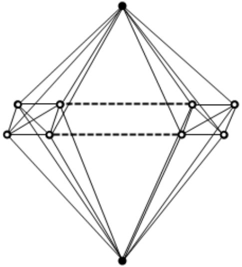

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- Proven they are Maxwell, conjectured that they are flexible
- Symmetry is lost when $\frac{d}{2}$ edges are added, so rigidity matrix analysis is more delicate

Thanks for listening!

Questions?

Thanks to my advisors Audrey Lee-St. John and Jessica Sidman for their guidance on my research.

Also, thanks to the Michigan REU and Patrick Boland for hosting/advising the REU program which I participated in last summer.