Edge-Disjoint Spanning Trees and Inductive Constructions

Laura Gioco

Fairfield University

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A 3D body-and-bar structure is rigid $\iff$ the associated graph is the union of 6 edge-disjoint spanning trees. [Tay, 1984]
Definitions

A **cycle** is a graph with an equal number of vertices and edges whose vertices may be placed around a circle in such a way that two vertices are adjacent if and only if they appear consecutively along the circle. A graph with no cycle is called **acyclic**.

Example
Definitions

A graph $G$ is **connected** if for all $u, v \in V(G)$ there exists a path from $u$ to $v$. 

Example

![Graph example](image-url)
Definitions

* A graph $G$ is a **tree** if it is connected and acyclic.
A tree $T = (V_T, E_T)$ is a **spanning tree** of a graph $G = (V, E)$ if $V_T = V$. 
Definition

Let $G = (V, E)$ be a multigraph with no loops. $G$ is $(k, k)$-tight if there is a partition of the edge set such that there are $k$ edge-disjoint spanning trees.

Example
Previous Work

- 2D Rigidity, Henneberg 0 and 1 [Henneberg 1911] [Laman 1970]
- Body-and-Bar Rigidity using Henneberg operations inductively to form tight graphs [Tay 1991]
- Generalized Counting Characterizations [Haas 2002]
Inductive constructions are prevalent in Rigidity Theory because they are often useful in proofs.
Henneberg 0: Inductive construction builds a rigid structure one vertex at a time.

**Example**

graph $G = (V, E)$
new graph $G' = (V \cup v, E \cup e_1, e_2, \ldots, e_k)$
This manner of inductive construction preserves the property of $k$ edge-disjoint spanning trees.
single vertex

\[ k = 3 \]
existing graph

new vertex $v$

$k = 3$
Inductive Construction

existing graph \hspace{1cm} new vertex $v$

$k = 3$
existing graph

new vertex $v$

$k = 3$
Inductive Construction

existing graph

new vertex \( v \)

\[ k = 3 \]
Inductive Construction

new vertex

existing $G$

$e_1, e_2, \ldots, e_k$

new graph $G' = (V \cup v, E \cup e_1, e_2, \ldots, e_k)$

graph $G = (V, E)$
Body-and-CAD Rigidity- recent results for 2D

- Combinatorial Characteristics of 2D Body-and-CAD Rigidity: associated graph $G = (V, S \cup D)$, exists $S' \subseteq S$ such that $S \setminus S'$ is edge-disjoint union of 2 spanning trees and $D \cup S'$ is edge-disjoint union of 1 spanning tree [Sidman, Lee-St. John, LaMon 2012]

Example

- We study a new class of graphs that arise from rigidity of body-and-CAD structures.
Body-and-CAD Rigidity- recent results for 3D

- Combinatorial Characteristics of 3D Body-and-CAD Rigidity: associated graph $G = (V, S \cup D)$, exists $S' \subseteq S$ such that $S \setminus S'$ is edge-disjoint union of 3 spanning trees and $D \cup S'$ is edge-disjoint union of 3 spanning tree [Sidman, Lee-St. John 2012]
- this does not include point-point coincidence constraints
Generalizing Counts

- 2D: $2 + 1 = 3$
- 3D: $3 + 3 = 6$
- general: Let $a + g = k$, where $a$ is the number of solid trees, $g$ is the number of trees that might include dashed edges, and $k$ is the total number of spanning trees.
Generalizing the Construction

- can partition into $k = a + g$ spanning trees
- inductive constructions for $(k, k)$-tight graphs exist
- generalize inductive construction to take into account dashed and solid edges
Motivation

graph $G = (V, S \sqcup D)$

new graph $G' = (V \cup v, S \cup e_1, ..., e_n \sqcup D \cup e_{n+1}, ..., e_k)$
$k = 3, g = 2$

existing $G$

$e_1, e_2, e_3$

new vertex

graph $G = (V, S \sqcup D)$

new graph $G' = (V \cup v, S \cup e_2 \sqcup D \cup e_1 \cup e_3)$
Also \( k = 3, g = 2 \)

Graph \( G = (V, S \sqcup D) \)

New graph \( G' = (V \cup v, S \cup e_1 \cup e_2 \sqcup D \cup e_3) \)
Or even $k = 3, g = 2$

existing $G$

$e_1, e_2, e_3$

new vertex

graph $G = (V, S \sqcup D)$

new graph $G' = (V \cup v, S \cup e_1 \cup e_2 \cup e_3 \sqcup D)$
This implies that this constructive step creates a new graph that satisfies the $k = a + g$ spanning trees property.
There are some \((k,k)\)-tight graphs which my constructive operation cannot form.
Since my inductive construction does not create every \((k, k)\)-tight graph, I would like to modify my construction to also use the other Henneberg move which removes edges before adding the new vertex.
Questions?