Kulikov surfaces

by

Stephen Coughlan (UMass Amherst)

joint work with Tsz On Mario Chan (Universität Bayreuth)

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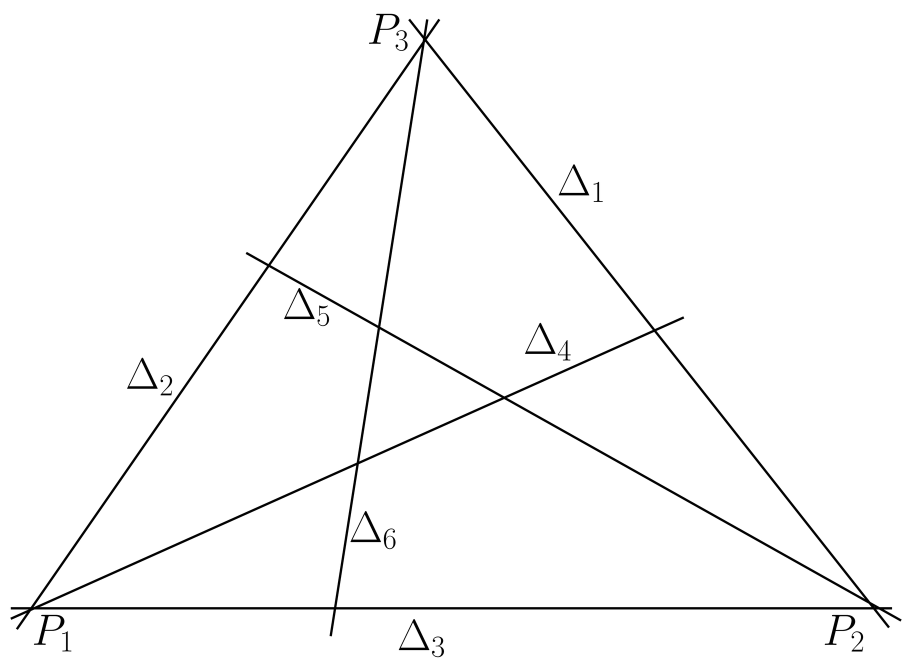
1 Introduction

I am interested in surfaces of general type with geometric genus $p_g = h^0(K_X) = 0$. Such surfaces have $1 \le K^2 \le 9$, and the fundamental group π_1 may be finite or infinite. In this poster, I describe joint work with Mario Chan where we construct a component of the moduli space corresponding to surfaces with $K^2 = 6$. Related examples were discovered by Burniat in the 1960's and from a different perspective by Inoue in the 1990's.

2 The Kulikov configuration

A Kulikov surface X is constructed by taking a particular nonsingular $(\mathbb{Z}/3)^2$-cover $\varphi\colon X \to \mathbb{P}^2$ branched over the configuration of lines displayed below.



The Kulikov configuration

The pertinent properties of this cover are governed by the group homomorphism

$$\Phi\colon \pi_1(\mathbb{P}^2 \smallsetminus \Delta) \to (\mathbb{Z}/3)^2.$$

Since this map factors through

$$H_1(\mathbb{P}^2 \smallsetminus \Delta, \mathbb{Z}) \cong \frac{\bigoplus_{i=1}^6 \mathbb{Z}\langle\Delta_i\rangle}{\Delta} \cong \mathbb{Z}^5,$$

we see that Φ is determined by the choices for $\Phi(\Delta_i)$.

Exercise Show that (up to choice of generators) there is a unique $(\mathbb{Z}/3)^5$-cover $\widehat{X} \to \mathbb{P}^2$ branched in the Kulikov line configuration.

The Kulikov surface is obtained by defining

$$\Phi = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$

where the ith column denotes the image of Δ_i.

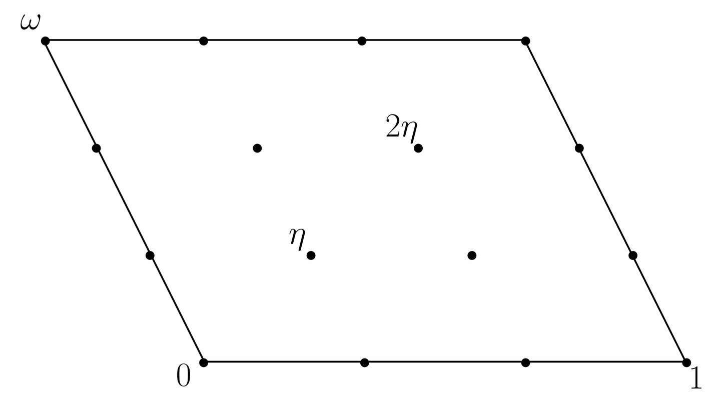
In fact, for X to be nonsingular we must blowup at the points P_i where three lines meet. Thus X is actually a $(\mathbb{Z}/3)^2$-cover of the del Pezzo surface Y of degree 6. Then since $K_X = \varphi^*(K_Y + \frac{2}{3}\sum_{i=1}^6 \Delta_i + \frac{2}{3}\sum_{j=1}^3 E_j)$ it is easy to check that K_X is ample and $K^2 = 6$. A similar calculation involving $\varphi_*(\omega_X)$ shows that $p_g(X) = 0$.

3 Fermat cubic curves

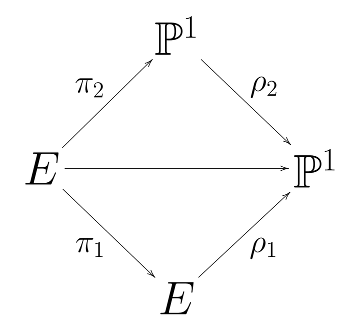
The plane cubic curve

$$C\colon (x^3 + y^3 + z^3 = 0) \subset \mathbb{P}^2$$

has lattice



The Fermat cubic lattice



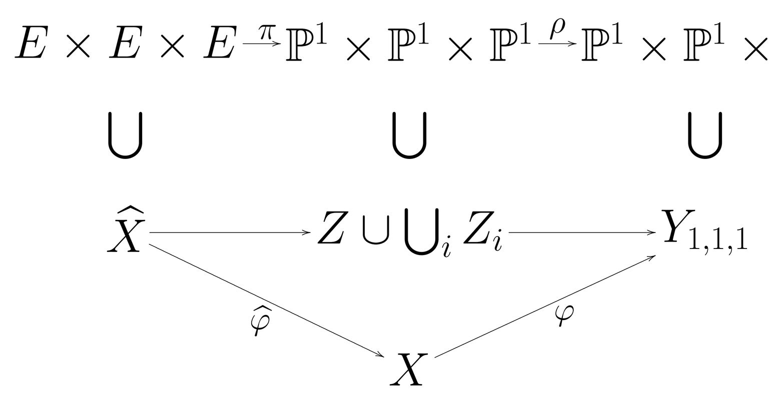
The $(\mathbb{Z}/3)^2$ action

Write $\omega = \exp(2\pi i/3)$ for the automorphism of order 3 on E and let η be a 3-torsion point fixed by ω. Then there is an action of $(\mathbb{Z}/3)^2 = \langle \eta, \omega\rangle$ on E which gives rise to the above diagram. The map π_1 is the isogeny induced by the translation group $\langle\eta\rangle$ and π_2 corresponds to $\langle\omega\rangle$ which fixes $0, \eta, 2\eta$. The maps ρ_1 and ρ_2 are the $\mathbb{Z}/3$-quotients induced by the appropriate factor groups.

Exercise Write down the action of $(\mathbb{Z}/3)^2$ in terms of coordinates on the plane cubic.

4 Construction in the style of Inoue

Take the direct product of three copies of the above diagram to get the first row of the following picture



Consider the pencil of degree six del Pezzo surfaces $Y \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ given by the equation of tridegree $(1,1,1)$

$$\lambda x_1 x_2 x_3 = \mu y_1 y_2 y_3.$$

Then $\rho^{-1}(Y)$ splits into three components Z_i, $i = 0, 1, 2$, each of which has stabiliser isomorphic to $(\mathbb{Z}/3)^2$. Fix $Z = Z_0$ and define $\widehat{X} = \pi^{-1}(Z)$. Thus $\widehat{X}$ is a hypersurface in $E \times E \times E$ of tridegree $(3,3,3)$. The quotient of $\widehat{X}$ by a certain subgroup $G \cong (\mathbb{Z}/3)^3$ gives a Kulikov surface X. Note that $\widehat{X}$ is the maximal $(\mathbb{Z}/3)^5$-cover from the exercise.

5 A connected component of the moduli space

We get a nice description of the topological fundamental group $\pi_1(X)$ via the short exact sequence

$$0 \to \Lambda^3 \to \pi_1(X) \to (\mathbb{Z}/3)^3 \to 0,$$

where Λ is the Fermat lattice.

Using these two descriptions of Kulikov surfaces and the structure of the fundamental group, we prove

Theorem 1. *The pencil of Kulikov surfaces forms a 1-dimensional connected component of the moduli space.*

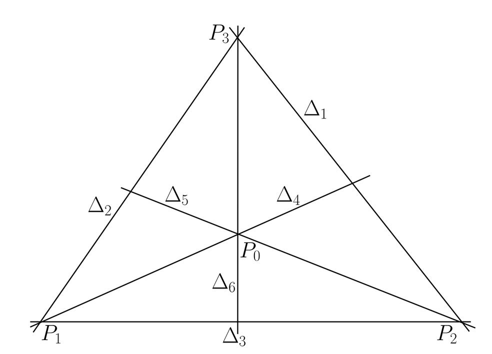
Other fun results:

Theorem 2. *The bicanonical map of a Kulikov surface X is a birational morphism.*

Theorem 3. *Kulikov surfaces verify the Bloch conjecture: $A_0^0(X) = 0$, where A_0^0 denotes the zero-cycles of degree 0.*

6 Some degenerations

For special elements of the pencil of del Pezzo surfaces, we get degenerate Kulikov surfaces.



If $\lambda = \mu$ the branch locus becomes the complete quadrangle. Then the $(\mathbb{Z}/3)^2$-cover has an elliptic singularity of degree 9 over P_0. After resolving, the minimal model is the hyperelliptic surface $(E \times E)/(\mathbb{Z}/3)^2$.

If $\lambda = 0$ or $\mu = 0$ the del Pezzo surface breaks into three pieces, indicated by dotted lines in the picture. The $(\mathbb{Z}/3)^2$-cover has orbifold normal crossing singularities

$$(xy = 0) \subset \frac{1}{3}(1,2,1)$$

at the points where Δ_i and Δ_i' intersect, for $i = 4, 5, 6$.