Snc Fano varieties and log Fano manifolds

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0 Abstract

We classify n-dim'l log Fano mfds (X, D) with index $r \geq n/2$, $\rho(X) \geq 2$ and $D \neq 0$.

1 (Log) Fano manifold

Fano manifolds play essential rolls in algebraic geometry, especially in birational geometry. We also introduce the notion of log Fano manifold.

Definition (X, D) is an n-dim'l **log Fano manifold** if

- X is smooth projective variety $/\mathbb{C}$,
- D is a reduced snc divisor on X and
- $-K_X - D$ is ample.

Remark Log Fano mfds have been considered by Hironobu Maeda. He classified the case $D \neq 0$, $n \leq 3$.

We are interested in the classification problem of (log) Fano manifolds, especially in its "geographical" properties.

Definition The **index** r of (X, D) is

$$r = \max\{r' \in \mathbb{Z}_{>0} \mid -K_X - D \sim r'L \ (\exists L : \text{ divisor})\}.$$

Known results X is classified if:

(1) $r \geq n-2$, or

(2) $r > n/2$ and $\rho(X) \geq 2$, where $\rho(X)$ is the Picard number of X (Wiśniewski's result);

(a) If $r \geq (n+2)/2$, then $X \simeq \mathbb{P}^{r-1} \times \mathbb{P}^{r-1}$.

(b) If $r = (n+1)/2$, then $X \simeq \mathbb{P}^{r-1} \times \mathbb{Q}^r$, $\mathbb{P}_{\mathbb{P}^r}(T_{\mathbb{P}^r})$ or $\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}^{\oplus r-1} \oplus \mathcal{O}(1))$.

Related to (2a), there is a very famous conjecture;

Conjecture (Mukai Conjecture M^n_ρ)
Fix n, $\rho \geq 1$. If X is an n-dim'l Fano mfd of index r with $\rho(X) \geq \rho$ and $r \geq (n+\rho)/\rho$, then $X \simeq (\mathbb{P}^{r-1})^\rho$.

- M^n_ρ is true if $n \leq 5$ or $\rho \leq 3$.

We also introduce a "log version" of this conjecture:

Conjecture (Log Mukai Conjecture LM^n_ρ)
Fix n, $\rho \geq 2$. If (X, D) is an n-dim'l log Fano mfd of index r and $D \neq 0$ with $\rho(X) \geq \rho$ and $r \geq (n+\rho-1)/\rho$, then $X \simeq \mathbb{P}_{(\mathbb{P}^{r-1})^{\rho-1}}(\mathcal{O}^{\oplus r} \oplus \mathcal{O}(m_1, \dots, m_{\rho-1}))$, $D \simeq \mathbb{P}_{(\mathbb{P}^{r-1})^{\rho-1}}(\mathcal{O}^{\oplus r})$ with $m_1, \dots, m_{\rho-1} \geq 0$.

- LM^2_ρ is true.

2 Snc Fano variety

We consider *snc (simple normal crossing) Fano varieties*, which is very natural generalizations of Fano mfds.

Definition $\mathcal{X}$ is an n-dim'l **snc Fano variety** if

- conn. proj. scheme $/\mathbb{C}$,
- $\hat{\mathcal{O}}_{\mathcal{X},x} \simeq \mathbb{C}[[t_1, \dots, t_{n+1}]]/(t_1 \cdot \dots \cdot t_k)$ $(\forall x \in \mathcal{X})$,
- $\forall X \subset \mathcal{X}$ irr. cpnt is smth and
- the anti-dualizing sheaf $\omega^\vee_{\mathcal{X}}$ is ample.

We also define the *index* r of $\mathcal{X}$

$$r = \max\{r' \in \mathbb{Z}_{>0} \mid \omega^\vee_{\mathcal{X}} \simeq \mathcal{L}^{\otimes r'} \ (\exists \mathcal{L} \in \mathrm{Pic}(\mathcal{X}))\}.$$

Many applications are expected, as with Fano manifolds.

Ex. of application (Kollár) For $\mathcal{X}$ with $\omega^\vee_{\mathcal{X}} \simeq \mathcal{L}^{\otimes r}$ and $r \geq 2$, there exists $(0 \in Z)$ a germ of $(n+1)$-dim'l terminal sing. with a partial resol. $(\mathcal{X} \subset W) \to (0 \in Z)$;

- W: canonical sing.,
- $\mathcal{X} \subset W$ is a Cartier divisor; $\mathcal{N}_{\mathcal{X}/W} \simeq \mathcal{L}^\vee$,
- K_Z is a Cartier divisor and $Z \setminus \{0\}$ is simply connected.

Remark Any irr. cpnt with conductor $(X, D) \subset \mathcal{X}$ is a log Fano mfd. Conversely, we can construct $\mathcal{X}$ from log Fano mfds under certain 'gluing condition'.

3 Main theorems

Main Theorem A
For (X, D) with $D \neq 0$, either of the following holds:
(1) $\mathrm{Pic}(X) \to \mathrm{Pic}(D)$ is injective, or
(2) $\exists \pi \colon X \to Y$ $\mathbb{P}^1$-bdle structure and D is a section of π.

Main Theorem B

$$\mathrm{M}^n_\rho + \mathrm{LM}^n_\rho \Rightarrow \mathrm{LM}^{n+1}_\rho.$$

Main Theorem C
We have classified (X, D) with $r \geq n/2$, $\rho(X) \geq 2$ and $D \neq 0$.

- If $r > n/2$, then it is nothing but LM^n_2.
- If $r = n/2$ and $r \geq 2$, then (X, D) is:

No.	X	D
A	$\mathrm{Bl}_{\mathbb{P}^{r-2}} \mathbb{P}^{2r}$	$\mathrm{Bl}_{\mathbb{P}^{r-2}} \mathbb{P}^{2r-1}$ with $\mathbb{P}^{r-2} \subset \mathbb{P}^{2r-1} \subset \mathbb{P}^{2r}$ linear
B	$\mathbb{P}^{r-1} \times \mathbb{P}^{r+1}$	$\mathbb{P}^{r-1} \times \mathbb{Q}^r$
		$\mathbb{P}^{r-1} \times \mathbb{P}^r \cup \mathbb{P}^{r-1} \times \mathbb{P}^r$
C	$\mathbb{P}_{\mathbb{P}^{r-1}}(\mathcal{O}^{\oplus r} \oplus \mathcal{O}(m_1) \oplus \mathcal{O}(m_2))$ with $0 \leq m_1 \leq m_2, 1 \leq m_2$	$\mathbb{P}_{\mathbb{P}^{r-1}}(\mathcal{O}^{\oplus r} \oplus \mathcal{O}(m_1))$ $\cup \mathbb{P}_{\mathbb{P}^{r-1}}(\mathcal{O}^{\oplus r} \oplus \mathcal{O}(m_2))$
D	the double cover of $\mathbb{P}_{\mathbb{P}^{r-1}}(\mathcal{O}^{\oplus r+1} \oplus \mathcal{O}(m))$ $(m \geq 0)$ with the smooth branch $B \in \lvert\mathcal{O}_{\mathbb{P}}(2)\rvert$	the strict transform of $\mathbb{P}_{\mathbb{P}^{r-1}}(\mathcal{O}^{\oplus r+1})$ $(\simeq \mathbb{P}^{r-1} \times \mathbb{Q}^r)$; smooth
E	$(r \geq 3)$ $\mathbb{P}_{\mathbb{Q}^r}(\mathcal{O}^{\oplus r} \oplus \mathcal{O}(m))$ with $m \geq 0$	$\mathbb{P}_{\mathbb{Q}^r}(\mathcal{O}^{\oplus r})(\simeq \mathbb{P}^{r-1} \times \mathbb{Q}^r)$
F	$(r = 2)$ $\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(m_1, m_2))$ with $0 \leq m_1 \leq m_2$	$\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}^{\oplus 2})(\simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$
G	$\mathbb{P}_{\mathbb{P}^r}(T_{\mathbb{P}^r} \oplus \mathcal{O}(m))$ with $m \geq 1$	$\mathbb{P}_{\mathbb{P}^r}(T_{\mathbb{P}^r})$
H	$\mathbb{P}^r \times \mathbb{P}^r$	$\mathbb{P}_{\mathbb{P}^r}(T_{\mathbb{P}^r}) \in \lvert\mathcal{O}(1,1)\rvert$
		$\mathbb{P}^{r-1} \times \mathbb{P}^r \cup \mathbb{P}^r \times \mathbb{P}^{r-1}$
I	$\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}^{\oplus r} \oplus \mathcal{O}(1))$	$\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}^{\oplus r-1} \oplus \mathcal{O}(1))(\simeq \mathrm{Bl}_{\mathbb{P}^{r-2}} \mathbb{P}^{2r-1})$
		$\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}^{\oplus r}) \cup \mathbb{P}_{\mathbb{P}^{r-1}}(\mathcal{O}^{\oplus r} \oplus \mathcal{O}(1))$
J	$\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}^{\oplus r-1} \oplus \mathcal{O}(1) \oplus \mathcal{O}(m))$ with $m \geq 1$	$\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}^{\oplus r-1} \oplus \mathcal{O}(1))(\simeq \mathrm{Bl}_{\mathbb{P}^{r-2}} \mathbb{P}^{2r-1})$
K	$\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}^{\oplus r} \oplus \mathcal{O}(m))$ with $m \geq 2$	$\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}^{\oplus r}) \cup \mathbb{P}_{\mathbb{P}^{r-1}}(\mathcal{O}^{\oplus r} \oplus \mathcal{O}(m))$

Corollary
We have classified (X, D) with $r \geq n-2$.

4 Idea of the proofs

Sketch of Theorem A

(1) Consider a $(-D)$-MMP which is also a $(K_X + D)$-MMP which ends with a fiber type contraction

$$X = X^0 \dashrightarrow X^1 \dashrightarrow \cdots \dashrightarrow X^k \to Y.$$

(2) $\dim \mathrm{Coker}(\mathrm{N}_1(D^i) \to \mathrm{N}_1(X^i))$ is constant.

(3) Unless $k = 0$ and $X^k \to Y$ is a $\mathbb{P}^1$-bundle, $\mathrm{N}_1(D^k) \to \mathrm{N}_1(X^k)$ is surjective.

Sketch of Theorems B and C

Pick the ext'l contr. $\pi \colon X \to Y$ w.r.t. an ext'l ray $R \subset \mathrm{NE}(X)$ such that $(D \cdot R) > 0$ and see π in detail. (The morphism is very 'special' since the length of R is long.)