Degenerations of surfaces and vector bundles

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Abstract

A recent construction of Hacking relates the classification of stable vector bundles on a surface of general type with $p_g = 0$ and the boundary of the moduli space of deformations of the surface. We analyze this correspondence for rank 2 and $K^2 = 1$.

1. Hacking's construction

Let $a < n$ be coprime positive integers. A Wahl singularity of type $\frac{1}{n^2}(1, na-1)$ is a cyclic quotient singularity $0 \in (\mathbb{C}^2/(\mathbb{Z}/n\mathbb{Z}))$ given by $\mathbb{Z}/n\mathbb{Z} \supset 1 : (u,v) \mapsto (\zeta u, \zeta^{an-1}v)$, where $\zeta = exp(2\pi i/n^2)$.

An exceptional vector bundle F on Y is a holomorphic vector bundle such that $\mathrm{Hom}(F,F) = \mathbb{C}$ and $\mathrm{Ext}^1(F,F) = \mathrm{Ext}^2(F,F) = 0$, in particular F is indecomposable, rigid, and unobstructed in families.

Theorem 1 (Hacking)**.** *Given a degeneration of a smooth surface Y to a surface X with a unique $\frac{1}{n^2}(1, na-1)$ singularity such that K_X is ample, one can construct an exceptional vector bundle F on Y of rank n with $c_1(F).K_Y = \pm a \mod n$ and such that F is stable with respect to K_Y.*

Let Y be a smooth projective surface of general type. We can define

$$S = \left\{ \frac{c_1(F)}{\mathrm{rank}\,(F)} \in H^2(Y,\mathbb{Q}) \middle| \begin{array}{l} F \text{ is an exceptional vector bundle} \\ \text{of rank } n, \text{ stable with respect to} \\ K_Y,\ c_1(F).K \text{ is coprime to } \mathrm{rank}\,F. \end{array} \right\} \Big/ \sim,$$

where the equivalence relation is generated by translation by elements of $H^2(Y,\mathbb{Z})$, multiplication by ± 1, and the monodromy group.

Thus we can consider a correspondence

$$\Phi\colon \left\{ \begin{array}{l} \text{boundary components of } \overline{\mathcal{M}} \text{ corresponding to the} \\ \text{degeneraions } Y \rightsquigarrow X, \text{ where X has a unique} \\ \frac{1}{n^2}(1, na-1) \text{ singularity} \end{array} \right\} \longrightarrow S. \tag{1}$$

where $\overline{\mathcal{M}}$ is the compactification of the moduli space $\mathcal{M}$ of surfaces wit the same numerical data as Y.

The analogous correspondence is known to be surjective for all del Pezzo surfaces.

Aim: We study the correspondence (1) for Godeaux surfaces Y in the case $n = 2$. In this case Y degenerates to a surface with a unique $\frac{1}{4}(1,1)$ singularity.

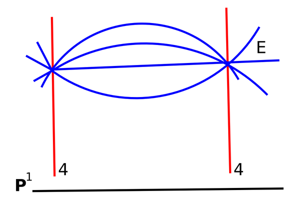
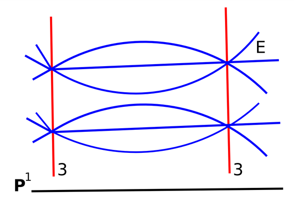
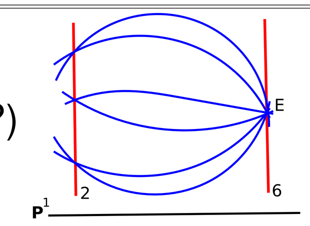
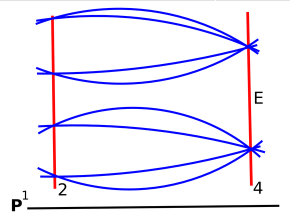
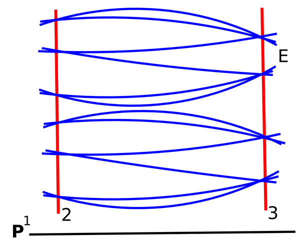
A *numerical Godeaux surface* is a minimal surface Y of general type with $K^2 = 1$ and $p_g = q = 0$. Godeaux surfaces are categorized by $H_1(Y)$. Reid proved that $H_1(Y)$ could be $\mathbb{Z}/5\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$ or trivial.

Godeaux surfaces with $H_1(Y) = \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}$, and $\mathbb{Z}/3\mathbb{Z}$ are completely classified by Reid. Some examples of Godeaux surfaces with $H_1(Y) = \mathbb{Z}/2\mathbb{Z}$ and 0 are known, but there is no complete classification of such surfaces.

2. Summary of results

Let Y be a smooth Godeaux surface, and let X be a degeneration of Y such that X has only a unique $\frac{1}{4}(1,1)$ singularity. Denote by $\widetilde{X}$ a minimal resolution of this singularity.

Assume that $H_1(X) \simeq H_1(Y)$. We obtain the following correspondence between the degenerations $Y \rightsquigarrow X$ of a smooth Godeaux surface to the surface with unique $\frac{1}{4}(1,1)$ singularity and the exceptional vector bundles of rank 2 on Y.

$H_1(Y)$	$\widetilde{X}$ corresponding to the degeneration $Y \rightsquigarrow X$	$c_1(F)$ of an exceptional vector bundle F
$\mathbb{Z}/5\mathbb{Z}$	none	$c_1(F) = 3K - A_1 - A_2$, $A_i^2 = -2$
$\mathbb{Z}/4\mathbb{Z}$		$c_1(F) = K + \sigma$
	none	$c_1(F) = 3K - A_1 - A_2$
$\mathbb{Z}/3\mathbb{Z}$		$c_1(F) \sim 3K - A_1 - A_2$
$\mathbb{Z}/2\mathbb{Z}$	(?) 	$c_1(F) = K + \sigma$
		$c_1(F) \sim 3K - A_1 - A_2$
$\{0\}$		$c_1(F) \sim 3K + \alpha$, $\alpha^2 = -4$

We see that the correspondence (1) is not surjective for the Godeaux surfaces with $H_1(Y) = \mathbb{Z}/5\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$. We expect the correspondence to be bijective in the cases $H_1(Y) = \mathbb{Z}/3\mathbb{Z},\ \mathbb{Z}/2\mathbb{Z},\ 0$.

3. Degenerations of Godeaux Surfaces

Theorem 2. *Let Y be a smooth Godeaux surface and let X be its degeneration such that X has a unique $\frac{1}{4}(1,1)$ singularity. If such a degeneration X exists, then the minimal resolution of the singularity $\widetilde{X}$ is a minimal properly elliptic surface with two multiple fibers of multiplicities m_1 and m_2, and $H_1(\widetilde{X}) = \mathbb{Z}/(m_1, m_2)\mathbb{Z}$.*

Let E be the (-4) – curve on $\widetilde{X}$ obtained by the resolution of the $\frac{1}{4}(1,1)$ singularity on X, and let F be a general fiber of the fibration $\widetilde{X}$. Set $n := E.F$, then $m_i|n$ for $i = 1, 2$. Moreover, there are only five possible choices for m_i and n, namely (a) $m_1 = m_2 = n = 4$; (b) $m_1 = m_2 = 3$, $n = 6$; (c) $m_1 = 2$, $m_2 = 6$, $n = 6$; (d) $m_1 = 2$, $m_2 = 4$, $n = 8$; (e) $m_1 = 2$, $m_2 = 3$, $n = 12$.

Sketch of the Proof.

1) $K^2_{\widetilde{X}} = 0$ and $p_g(\widetilde{X}) = 0$.

2) $\widetilde{X}$ can not be a rational surface, Enriques surface or a surface of general type

3) So $\widetilde{X}$ must be a properly elliptic surface. Thus by the Kodaira canonical bundle formula $K_{\widetilde{X}} \sim -1 + \sum(1 - \frac{1}{m_i})F$, where F is a general fiber of the fibration

4) Using the fact that $E.K_{\widetilde{X}} = 2$ and the possibilities for $H_1(Y)$ we conclude that $\widetilde{X}$ has exactly 2 multiple fibers satisfying $\frac{1}{m_1} + \frac{1}{m_2} + \frac{2}{n} = 1$. This equation has only five integer solutions satisfying the condition that $m_i|n$.

Remark 1. We know that degenerations corresponding to the cases (a) and (b) can happen by explicit equations, and (d) and (e) because of the constructions of Y. Lee and J. Park. We expect that case (c) is also possible, but we do not have a construction.

4. Exceptional vector bundles

We are looking for an exceptional vector bundle F of rank 2 on a Godeaux surface Y such that $c_1(F).K_Y = 1 \mod 2$ and F is stable with respect to K_Y.

We use the following method of construction of vector bundles of rank 2:
After a twist by a line bundle ($F \mapsto M \otimes F$), any rank two vector bundle on a surface Y can be described as an extension

$$0 \to \mathcal{O}_Y \to F \to L \otimes \mathcal{I}_Z \to 0,$$

where

- $L = c_1(F)$ is a line bundle on the surface Y,
- Z is a zero dimensional subscheme of Y with $\#Z = c_2(F)$, and
- divisor $K_Y + L$ satisfies the Cayley-Bacharach property with respect to Z: *If a nonzero section $s \in H^0(K + L)$ vanishes on $Z' \subset Z$, where $\#Z' = \#Z - 1$, then s vanishes on Z.*

For an exceptional vector bundle F of rank two we have the following relation between $c_1(F)$ and $c_2(F)$:

$$c_2(F) = \frac{1}{4}(c_1(F)^2 + 3).$$

We expect monodromy group to be $\mathrm{Aut}(K^\perp) = W(E_8)$. Assuming this, up to the equivalence relation defined in section 1 and up to the torsion element of $H^2(Y)$, we have only two choices of $c_1(F)$: either $c_1(F) \sim K$ or $c_1(F) \sim 3K + \alpha$, where $\alpha \in K^\perp$ is some element satisfying $\alpha^2 = -4$.

Case $c_1(F) = K$

Theorem 3. *A vector bundle of rank 2 on a Godeaux surface Y such that $c_1(F) = K + \sigma$, where $\sigma \in \mathrm{Tors}\,H^2(Y)$ can be obtained from the following exact sequence*

$$0 \to \mathcal{O}_Y \to F \to \mathcal{O}(K + \sigma) \otimes \mathcal{I}_P \to 0,$$

where P is one of the four base points of $|2K + \sigma|$. Such a vector bundle F is exceptional if and only if $\sigma \neq 0$ but $2\sigma = 0$ in $H^2(Y)$. Also F is stable with respect to K_Y.

Remark 2. Note that based on the choice of the point P we obtain four non-isomorphic vector bundles with the same rank, c_1 and c_2.

Remark 3. In the case $H_1(Y) = \mathbb{Z}/4\mathbb{Z}$ we can also obtain these exceptional vector bundles using the Hacking's construction.

Case $c_1(F) = 3K - A_1 - A_2$

Theorem 4. *Let Y be a general Godeaux surface with $H_1(Y) = \mathbb{Z}/5\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z}$ containing two disjoint (-2)–curves A_1 and A_2 on it. A vector bundle of rank 2 on Y such that $c_1(F) = 3K - A_1 - A_2$ can be obtained from the following exact sequence*

$$0 \to \mathcal{O}_Y \to E \to \mathcal{O}(3K - A_1 - A_2) \otimes \mathcal{I}_Z \to 0,$$

where Z is one of the four degree two clusters which get contracted under the map $\phi_{|4K|}$. Such a vector bundle F is exceptional.

Remark 4. A vector bundle F as in the Theorem 4 is stable with respect to K_Y in the case $H_1(Y) = \mathbb{Z}/5\mathbb{Z}$.