Integration on GIT quotients by non-abelian groups

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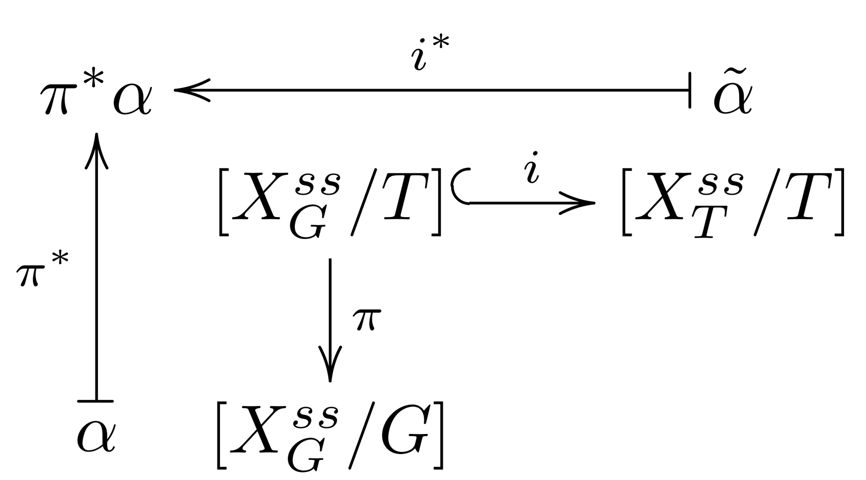
Set-up

For a reductive algebraic group G defined over a field k:

- Let $T < G$ be a fixed choice of split maximal torus.
- Let $(X, \mathcal{L})$ be a projective G-variety X with ample G-linearization $\mathcal{L}$.
- Let X_G^{ss} and X_G^s denote the open loci of GIT semistable and stable points in X with respect to the linearized action of G on the pair $(X, \mathcal{L})$.
- Assume $X_T^{ss} = X_T^s$ so that the quotient stack $[X_T^{ss}/T]$ is Deligne-Mumford with coarse moduli space $X/\!\!/T$, and similarly for G.

Fundamental construction

The goal of this study is to relate the integration of 0-dimensional Chow classes on the projective varieties $X/\!\!/G$ and $X/\!\!/T$, allowing one to reduce calculations for non-abelian group quotients to the abelian case. Given a 0-dimensional Chow class $\alpha \in A_0([X_G^{ss}/G])_{\mathbb{Q}}$, lift it to a class $\tilde{\alpha}$ with $i^*\tilde{\alpha} = \pi^*\alpha$:



Let $c \in A_*([X_T^{ss}/T])$ denote the top Chern class of the vector bundle $\mathfrak{g}/\mathfrak{t} \times_T X_T^{ss}$ on $[X_T^{ss}/T]$. We study the following ratio:

Definition

For α for which $\int_{X/\!\!/G} \alpha \neq 0$, the **GIT integration ratio** is defined to be

$$r_{G,T}^{X,\alpha} := \frac{\int_{X/\!\!/T}(c \frown \tilde{\alpha})}{\int_{X/\!\!/G} \alpha}.$$

Theorem

The ratio $r_G := r_{G,T}^{X,\alpha}$ is an invariant of the group G, independent of the choices of $T, X, \mathcal{L}, \alpha$, and $\tilde{\alpha}$.

A worked example

$V := k^{\oplus n}$, $G := PGL(V)$, $X := \mathbb{P}(\mathrm{End}(V))$, with linearized G-action induced by the left-multiplication of $SL(V)$ on $\mathrm{End}(V)$. Let T be the maximal torus of diagonal matrices. The GIT quotients and their Chow groups are:

- $X_G^{ss} = \{M : \det M \neq 0\} \Rightarrow [X_G^{ss}/G] = \operatorname{Spec} k$.
- $X_T^{ss} = \{M : \text{no column of } M \text{ is } 0\} \Rightarrow [X_T^{ss}/T] = (\mathbb{P}^{n-1})^{\times n}$.
- $A_*([X_G^{ss}/G])_{\mathbb{Q}} = \mathbb{Q}$.
- $A^*([X_T^{ss}/T])_{\mathbb{Q}} \cong \mathbb{Q}[t_1, \ldots, t_n]/(t_1^n, \ldots, t_n^n)$ (Künneth formula).

Integration on $X/\!\!/T$ and the class c can both be explicitly described:

- $\int_{X/\!\!/T} \prod_{i=1}^n t_i^{n-1} = 1$.
- $c = \prod_{i \neq j}(t_i - t_j)$.

Choosing $\alpha = 1$, we see that $r_{G,T}^{X,\alpha} = \int_{X/\!\!/T} c$ is the coefficient of $\prod_{i=1}^n t_i^{n-1}$ in the expansion of $\prod_{i\neq j}(t_i - t_j)$. Use the Vandermonde determinant to write $c = (-1)^{n(n-1)/2}(\det M_V)^2$, where

$$M_V := \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{bmatrix}.$$

Replace $\det M_V := \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n t_i^{\sigma(i)-1}$ and use the following rule for multiplication of monomials

$$\prod_{i=1}^n t_i^{\sigma(i)-1} \cdot \prod_{i=1}^n t_i^{\sigma'(i)-1} = \begin{cases} \prod_{i=1}^n t_i^{n-1} & : \sigma' = w_0 \circ \sigma; \\ 0 & : \text{ otherwise,} \end{cases}$$

where $w_0 := (1\ n) \cdots (\lfloor n/2 \rfloor\ \lfloor (n+1)/2 \rfloor) \in S_n$ is the longest element. This yields:

$$(\det M_V)^2 = \sum_{\sigma \in S_n} \operatorname{sgn}(w_0) \prod_{i=1}^n t_i^{n-1}.$$

Since $\operatorname{sgn} w_0 = (-1)^{n(n-1)/2}$, we conclude $r_{G,T}^{X,\alpha} = n!$.

Determining the value of r_G

It turns out r_G is functorial with respect to products and invariant under central extensions; we obtain the following corollary of the Theorem and our worked example:

Corollary

If G is, up to a central extension, the product $\prod_{i=1}^n PGL(k_i)$, then $r_G = |W|$, the order of the Weyl group.

Applying results from symplectic geometry

Using analytic techniques, Martin [2] calculates the analogous ratio r_G for G a compact Lie group acting via Hamiltonians on a closed symplectic manifold X, with the usual singular cohomology groups replacing the Chow groups. Using Seshadri's theory of relative GIT, Martin's result and the Theorem can be combined to prove the Main Theorem:

Main Theorem

If G is any reductive group defined over an arbitrary field k, then $r_G = |W|$, the order of the Weyl group.

Open Questions

- Can you find a purely algebraic proof of the Main Theorem, independent of Martin's result?
- What can be said, using Kirwan's partial desingularizations [1], in the case where $X_T^{ss} \neq X_T^s$?

References

[1] F. Kirwan. *Partial desingularisations of quotients of nonsingular varieties and their Betti numbers.* Ann. of Math. (2) (1985) vol. 122 (1) pp. 41-85.

[2] S. Martin. *Symplectic quotients by a nonabelian group and by its maximal torus.* arXiv (2000) vol. math.SG.