Cohomology ring of the Jacobi Factor of quasi-homogeneous plane curve singularities.

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joint work with Eugene Gorsky

Jacobi Factors

Let $x \in C \subset \mathbb{C}^2$ be a unibranched plane curve singularity. Let t be a normalizing parameter on C at x. Let $R \subset \mathbb{C}[[t]]$ be the complete local ring at x. Let $\delta = \dim \mathbb{C}[[t]]/R$. Since $x \in C$ is a plane curve singularity, it follows that $t^{2\delta}\mathbb{C}[[t]] \subset R$. Let $V = \mathbb{C}[[t]]/t^{2\delta}\mathbb{C}[[t]]$.

Definition. The *Jacobi factor* $\overline{JC}_x$ is the space of R-submodules $M \subset \mathbb{C}[[t]]$, such that $M \supset t^{2\delta}\mathbb{C}[[t]]$ and $\dim(\mathbb{C}[[t]]/M) = \delta$. In other words, $\overline{JC}_x$ is isomorphic to the subvariety of the Grassmannian $Gr(V, \delta)$, consisting of subspaces invariant under R-action.

A. Beauville showed in [1] that for a rational unibranched curve C its Compactified Jacobian is homeomorphic to the direct product of the Jacobi Factors $\overline{JC}_x$, $x \in Sing(C)$. J. Piontkowski proved that in some cases Jacobi factors admit algebraic cell decompositions ([11]; see also [7]). In [6] we further study Pointkowski's cell decompositions in the case of quasi-homogeneous plane curve singularities. We show that the cells of the decomposition can be enumerated by Young diagrams as follows:

Let $R^{m,n}$ be a $m \times n$-rectangle, where (m, n) are co-prime. Let $R^{m,n}_+ \subset R^{m,n}$ be the subset consisting of boxes which lie below the left-top to right-bottom diagonal. We show that the cells of the Jacobi factor of the singularity $\{x^m = y^n\}$ are in one-to-one correspondence with the Young diagrams contained in $R^{m,n}_+$. Moreover, if D is a diagram and C_D is the corresponding cell, then one can compute the dimension of the cell C_D from the combinatorics of D. Namely, one has

$$\dim C_D = \delta - h^+_{\frac{m}{n}}(D).$$

where $\delta = \frac{(n-1)(m-1)}{2}$, and $h^+_{\frac{m}{n}}$ is a statistic on Young diagrams, which arises from the Bialynicki-Birula cell decompositions of Hilbert schemes of points on $\mathbb{C}^2$. It is defined as follows:

Definition. ([8]) Let D be a Young diagram, $c \in D$. Let $a(c)$ and $l(c)$ denote the lengths of arm and leg for c. For each real nonnegative x define

$$h^+_x(D) = \#\left\{c \in D \Big| \frac{a(c)}{l(c)+1} \le x < \frac{a(c)+1}{l(c)}\right\}.$$

Thus we constructed a basis in the cohomology ring of the Jacobi factor, parametrized by the Young diagrams contained in the triangle $R^{m,n}_+$. Note, that in addition to the homological grading, which can be computed by the above formulas, one gets an additional statistic on the basis: the area of the corresponding diagram. Conjecturally, this correspond to a certain filtration on the cohomology ring, called perverse filtration ([9],[10]). We introduce the generating polynomials as follows:

$$C_{m,n}(q,t) = \sum_{D \subset R^{m,n}_+} q^{h^+(D)} t^{\delta - \text{area}(D)}$$

(here and forward we write h^+ instead of $h^+_{m,n}$, since m and n are obvious from the context.) If $n = m+1$, then this polynomial coincide with the (q,t)-Catalan number, introduced by A. Garsia and M. Haiman in [4]. Note also, that the dimension of the cohomology ring is equal to the number of Young diagrams in $R^{m,n}_+$, which is $\frac{1}{m+n}\binom{m+n}{n}$.

The Ring $A_{m,n}$

Following [3], we consider the quotient $A_{m,n}$ of the $\mathbb{C}[u_2, \ldots, u_m, v_2, \ldots, v_n]$ by the ideal generated by coefficients of the z-expansion of the equation

$$(1 + u_2z^2 + \ldots + u_mz^m)^n = (1 + v_2z^2 + \ldots + v_nz^n)^m.$$

There are several useful facts about this ring:

1. It is a 0-dimensional complete intersection of multiplicity $\frac{(m+n-1)!}{m!n!}$.
2. The equations are homogeneous in the following grading: $q(u_i) = q(v_i) = i$.
3. The variables v_i (or u_i) can be eliminated by writing

$$(1 + u_2z^2 + \ldots + u_mz^m)^{\frac{n}{m}} = (1 + v_2z^2 + \ldots + v_nz^n).$$

Therefore $A_{m,n}$ can be realized in the following way:

$$A_{m,n} = \mathbb{C}[u_2, \ldots, u_m]/(\text{Coef}_{\ge n+1}\left[(1 + u_2z^2 + \ldots + u_mz^m)^{\frac{n}{m}}\right]).$$

Although this reformulation is not (m,n)-symmetric, it is useful for the computations. In fact, one can show that the defining ideal is spanned by the coefficients at $z^{n+1}, \ldots, z^{m+n-1}$, and all further coefficients can be expressed through them.

The ring $A_{m,n}$ is equipped with the q-grading and the filtration by the powers of the maximal ideal. More precisely, for $f \in A_{m,n}$ we define

$$m(f) = \max\{k | f \in \mathfrak{m}^k\}.$$

It follows from [5] and [2] that $A_{m,n}$ carries a natural sl_2-action. Moreover, if $\{e, h, f\}$ is the standard basis of sl_2, then f acts by multiplication by u_2, and the action of h give rise to the q-grading.

Conjecture 1. *The cohomology ring of the Jacobi factor of the singularity $\{x^m = y^n\}$ is isomorphic to $A_{m,n}$.*

Conjecture 2. *There exists a monomial basis in $A_{m,n}$ such that*

1. *If a monomial ϕ is in the basis and another monomial ψ divides ϕ, then ψ is in the basis as well.*
2. *If $\phi = \prod u_i^{a_i}$ is in the basis, then $m(\phi) = \sum a_i$ (i.e. for any function f from the ideal $\deg(\phi + f) \le \deg(\phi)$).*
3. *For every basic function ϕ there exists a diagram $F(\phi) \subset R^{m,n}_+$, and this correspondence is bijective. Moreover, the following formulas hold:*

$$q(\phi) = |F(\phi)| + h^+(F(\phi)), \quad m(\phi) = h^+(F(\phi)).$$

4. *For a monomial ϕ from the basis with $q(\phi) \le \delta$, all the monomials $u_2^k\phi$ with $0 \le k \le \delta - q(\phi)$ also belong to the basis. Similarly, if $q(\phi) \ge \delta$, all the monomials $u_2^k\phi$ with $0 \ge k \ge \delta - q(\phi)$ are in the basis as well. Note that the primitive elements are exactly the monomials free of u_2.*

The last property means that the basis splits into bases of irreducible sl_2 sub-representations, and each such basis is of the form $\{\phi, u_2\phi, \ldots, u_2^{\delta - q(\phi)}\phi\}$, where ϕ is a monomial free of u_2.

The Case of $m = 3$

In this special case we can prove the Conjecture 2. Basically, since we have only two variables u_2 and u_3, the monomials are uniquely determined by the two gradings $q(u_2^au_3^b) = 2a + 3b$ and $m(u_2^au_3^b) = a + b$. Therefore, given a Young diagram D it is enough to know its area and h^+ to find the corresponding monomial.

Let $D \subset R^{3,n}_+$ be a diagram. It has not more than two rows. Suppose that the shorter row is of length α, and the longer is of length β. We have $0 \le \alpha \le \beta \le \frac{2n}{3}$, and $\alpha \le \frac{n}{3}$.

Let $n = 3k + r$, where $r = 1$ or 2. The map $G = F^{-1}$ assigning a monomial to each diagram D is piecewise linear with three domains of linearity:

1. $\beta \le k$. In this case one can check that $h^+(D_{\alpha,\beta}) = \beta$. On gets

$$G(D) = u_2^{\beta-\alpha}u_3^{\alpha}.$$

Remark that in this case we get all monomials $u_2^au_3^b$ with $a + b \le k$.

2. $\beta > k, \beta - \alpha \le k$. In this case one can check that $h^+(D_{\alpha,\beta}) = 2\beta - k$. One gets

$$G(D) = u_2^{3\beta-2k-\alpha}u_3^{\alpha-\beta+k}.$$

Remark that we get all monomials $u_2^au_3^b$ such that $a + b > k, 3b + a \le n - 1$, and $a + b + k$ even.

3. $\beta - \alpha > k$. In this case one can check that $h^+(D_{\alpha,\beta}) = 2\alpha + k + 1$. One gets

$$G(D) = u_2^{3\alpha-\beta+2k+2}u_3^{\beta-\alpha-k-1}.$$

Remark that we get all monomials $u_2^au_3^b$ such that $a + b > k, 3b + a \le n - 1$, and $a + b + k$ odd.

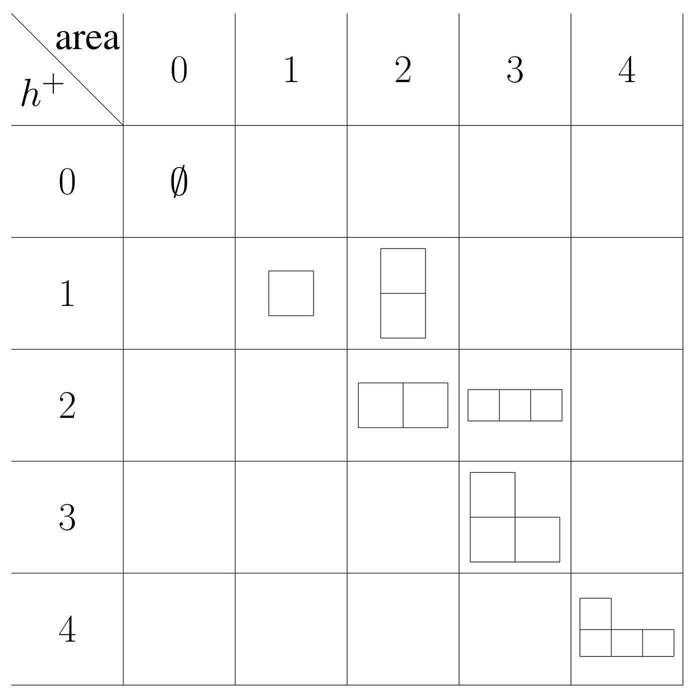
Combining all three cases, we see that the image of G is all the monomials $u_2^au_3^b$ with $a + 3b \le n - 1$

The following Theorem should be viewed as an evidence in support of the Conjecture 2:

Theorem. *The monomials $u_2^au_3^b$ with $a + 3b \le n - 1$ form a basis of the ring $A_{3,n}$.*

It is not hard to check that this basis satisfy all the properties from the Conjecture 2.

Example: $(m, n) = (3, 5)$.

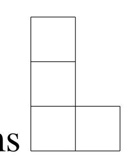
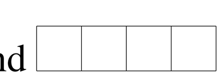


m \ $q-m$	0	1	2	3	4
0	1				
1		u_2	u_3		
2			u_2^2	u_2u_3	
3				u_2^3	
4					u_2^4

On the left we have the table of Young diagrams contained in $R^{3,5}_+$, organized with respect to the area (horizontal) and h^+ (vertical). On the right are the corresponding monomials.

Two diagrams with the same area and h^+

For $m > 3$ we have more than two variables. Therefore, gradings q and m do not determine the monomial. It turns out that on the combinatorial side we have the same problem: area and h^+ do not determine the diagram uniquely.

Let $m = 4$ and $n = 7$. The diagrams  and  has area $= 4$ and $h^+ = 2$.

Luckily, there are exactly 2 monomials in u_2, u_3, u_4 with $q = 6$ and $m = 2$: u_3^2 and u_2u_4.

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