Double MV Cycles and Loop-Group Crystal Combinatorics

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1 Introduction

I study how various geometric and algebraic constructions involving complex reductive groups can be extended to loop groups. A loop group is a central extension of the group of loops inside a complex reductive group. Loop groups also go by the name of **affine Kac-Moody groups**.

In this poster, we will examine loop versions of MV cycles (Double MV Cycles) and loop analogues of MV polytopes. A unifying thread in my research and in much of geometric representation theory is the notion of a **crystal**. We will see how the finite-dimensional theory of MV polytopes gives a wide unification of various crystal structures, and we will look at the conjectural analogue of this picture for loop groups.

2 Geometric Satake Correspondence

The main players in this poster are:

G	a complex reductive group (later a loop group)
T	a choice of maximal torus
N, N^-	a pair of opposite unipotent subgroups normalized by T
$\mathcal{K} = \mathbb{C}((t))$	formal Laurent series in one variable
$\mathcal{O} = \mathbb{C}[[t]]$	formal Taylor series in one variable
$\mathcal{G}r = G(\mathcal{K})/G(\mathcal{O})$	the affine Grassmannian for G (viewed as an ind-scheme)
$B(\infty)$	crystal associated to the negative part of the enveloping algebra

The following theorem provides a deep connection between the geometry of the affine Grassmannian and representation theory:

Theorem 1. *Geometric Satake [Ginzburg, Mirković-Vilonen] The category of $G(\mathcal{O})$-equivariant perverse sheaves on $\mathcal{G}r$ is equivalent as a symmetric monoidal category to the category of representations of the Langlands dual group of G.*

3 MV Cycles and MV Polytopes

The Mirković-Vilonen proof of the Geometric Satake Correpsondence provides finer information than just an equivalence of categories. In each representation they provide a basis, which is indexed by MV (Mirković-Vilonen) cycles, certain irreducible subvarieties of $\mathcal{G}r$. These varieties are defined to be the irreducible components of varieties obtained by intersecting $N(\mathcal{K})$-orbits and $N^-(\mathcal{K})$-orbits. Although, it is easy to parameterize these intersections, it is a non-trivial problem to explicitly describe the irreducible components.

Kamnitzer solved this problem by introducing **MV polytopes**. The MV polytope of an MV cycle is defined to be the image under the moment map for the torus action on $\mathcal{G}r$. Kamnitzer proved that each MV cycle is determined by its MV polytope, and gave explicit formulas for when a polytope is an MV polytope.

4 Double MV Cycles

Let us upgrade G to be a loop group. Then we can still make sense of $\mathcal{G}r$ as a set, but it won't have an ind-scheme structure, and there is currently no analogue of the Geometric Satake Correspondence. However, there is a general definition of MV cycles, due to Braverman-Finkelberg-Gaitsgory, which makes sense for any Kac-Moody group. Morever, they define a $B(\infty)$ crystal structure that makes sense in the setting of **double MV cycles** (MV cycles for loop groups).

Unfortunately, Kamnitzer's method for understanding MV cycles does not carry over to double MV cycles. His method amounts to studying the orders of poles that arise when an MV cycle acts on extremal weight vectors in fundamental representations. However, this data is not enough to distinguish double MV cycles.

In type A, we can try something else. The loop group will act on Fermionic Fock space, and we can ask whether this action can distinguish different double MV cycles. Indeed, this data is sufficient, and the combinatorial data extracted exactly matches the **Naito-Sagaki-Saito crystal**.

Theorem 2. *[-] In type A, one can distinguish double MV cycles by computing the orders of poles of the action on the Fock space basis. Colllecting such data gives us an isomorphism between the Braverman-Finkelberg-Gaitsgory crystal on double MV cycles and the Naito-Sagaki-Saito crystal.*

In this way, we can offer a new geometric proof that the Naito-Sagaki-Saito crystal is the $B(\infty)$ crystal.

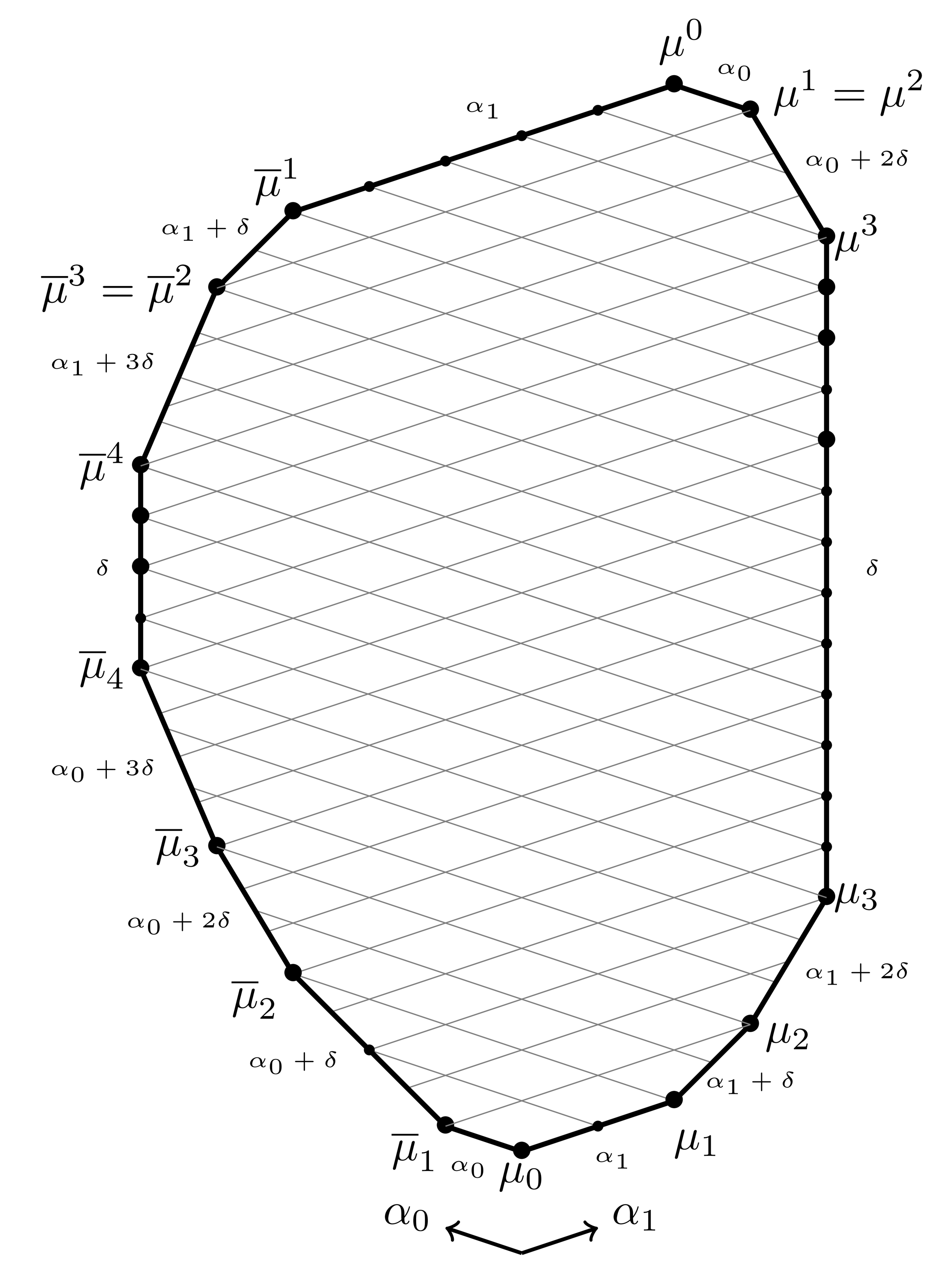
5 The Current State of Affairs

Realizations of $B(\infty)$ crystal	**Geometric**	**Combinatorial**	**Algebraic**
For Complex Reductive Groups	MV Cycles; Components of Quiver Varieties	MV Polytopes	Lusztig's Canonical Basis
For Loop Groups (Affine Groups)	Double MV Cycles; Components of Quiver Varieties	Affine MV Polytopes; Naito-Sagaki-Saito Crystal	Lusztig's Canonical Basis

Currently each of the objects in the first row (for complex reductive groups) give rise to MV polytopes via the work of Kamnitzer, Baumann-Kamnitzer, Berenstein-Fomin-Zelevinsky, Lusztig.

For loop groups, the situation is far from complete. We summarize the work that has been accomplished:

- In this poster, we have stated a connection between type A double MV cycles and the Naito-Sagaki-Saito crystal.
- Baumann-Kamnitzer-Tingley explain how to explicitly extract certain polytopes from components of quiver varieties. In this way they give a definition of **affine MV polytopes**
- There is another definition of affine MV polytopes associated to a canonical basis element coming from the PBW bases of Beck-Chari-Pressley, Akasaka, and Beck-Nakajima.
- In affine rank 2, Baumann-Dunlap-Kamnitzer-Tingley give an explicit combinatorial model for affine MV polytopes.
- Work to appear by the author and Tingley shows that all three notions of affine MV polytope agree in affine rank 2.
- It is still a wide-open problem to extract affine MV polytopes from double MV cycles.



An affine MV polytope for $\widehat{\mathfrak{sl}}_2$. Picture courtesy of Peter Tingley.