Boundary Divisors in the Moduli Space of Stable Quintic Surfaces

Julie Rana

University of Massachusetts, Amherst

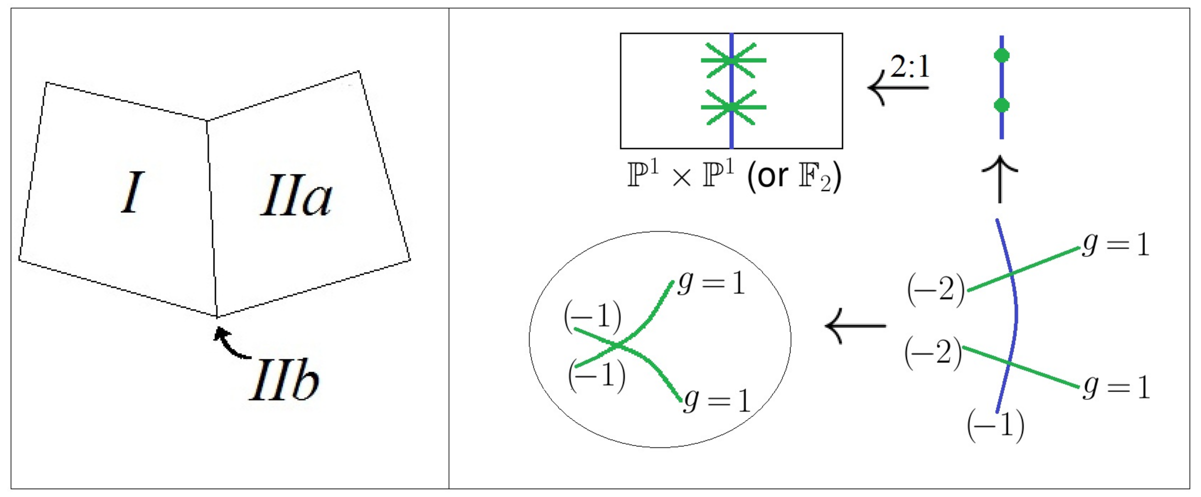
1 Introduction

The moduli space $\mathcal{M}_{K^2,\chi}$ of minimal surfaces with fixed topological invariants, ample canonical class and at most du Val singularities (Gieseker '77) admits a compactification $\overline{\mathcal{M}}_{K^2,\chi}$, the moduli space of stable surfaces (Kollár-Shepherd-Barron '88). A stable surface is a connected projective surface with ample canonical class and semi log canonical singularities. In general, $\overline{\mathcal{M}}_{K^2,\chi}$ can be arbitrarily complicated and singular (Catanese '86, Vakil '06).

Our goal is to describe some of the boundary divisors in $\overline{\mathcal{M}}_{K^2,\chi}$. There are two natural loci in the moduli space which are Cartier divisors if certain conditions are met. One of these corresponds to normal surfaces which have a unique Wahl singularity. A different set of expected boundary divisors corresponds to surfaces with orbifold normal crossings with some conditions on the orbifold normal bundle. We discuss these surfaces in the case of stable quintics.

2 Numerical Quintic Surfaces

$\mathcal{M}_{5,5}$ is the moduli space of surfaces with invariants $K^2 = 5$, $p_g = 4$, $q = 0$.

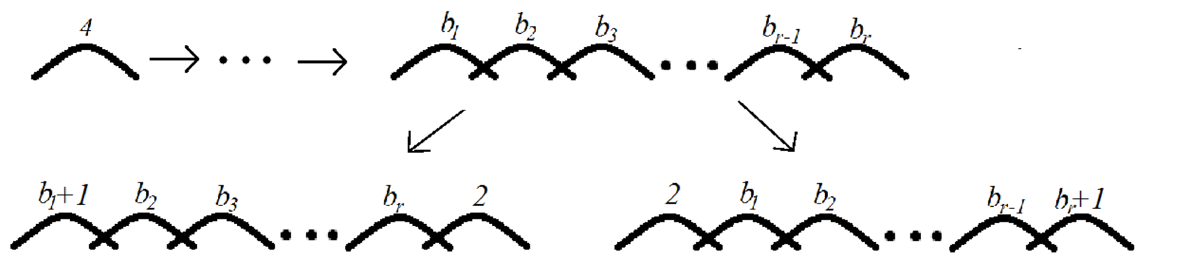


On left, $\mathcal{M}_{5,5}$. Components I and IIa are 40-dimensional; IIb is 39-dimensional (Horikawa '75). On right, how to obtain a surface of type IIa from a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ or of type IIb from a double cover of $\mathbb{F}_2$.

3 Wahl Singularities

Definition. A *Wahl singularity* is a $\frac{1}{n^2}(1, na-1)$ cyclic quotient singularity where $(a, n) = 1$.

The resolution of a Wahl singularity contains a string of exceptional curves $C_1, \ldots, C_r$ called a T*-string*. Every T-string of length $r+1$ can be obtained from a T-string of length r as in the following figure (Wahl '81):



How to obtain a T-string of length $r+1$ from a T-string of length r. Here, $C_i^2 = -b_i$.

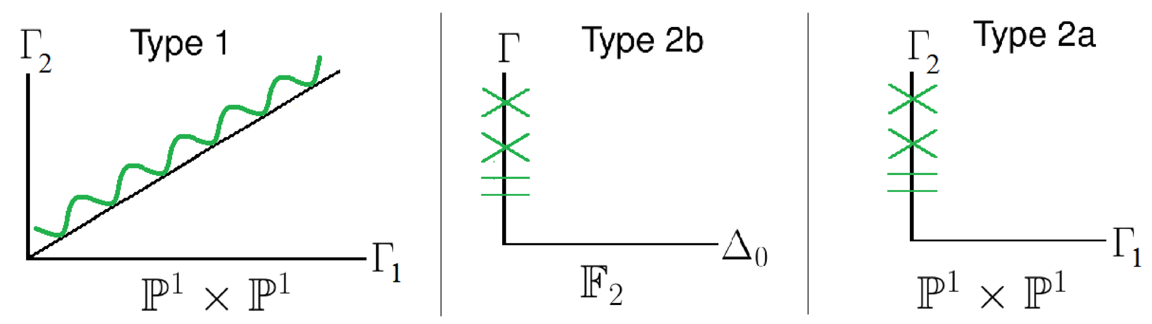
Wahl singularities are log terminal and admit a one-parameter $\mathbb{Q}$-Gorenstein smoothing given by $(xy = z^n + t) \subset \frac{1}{n}(1, -1, a) \times \mathbb{C}[t]$. Given a stable surface X with a unique Wahl singularity, its equisingular deformations will correspond to a divisor in $\overline{\mathcal{M}}_{K^2,\chi}$ as long as its smoothing globalizes to all of X.

The first main result is the following:

Theorem. *Let X be a surface of general type with $K_X^2 = 2p_g - 3$, $p_g \geq 3$, and $q = 0$ (X sits one line above the Noether line) with a unique Wahl singularity at $p \in X$. Suppose $\tilde{X}$ is the minimal resolution of X and S is the minimal model of $\tilde{X}$. If S is of general type then p is a $\frac{1}{4}(1,1)$ singularity.*

In the case of stable quintics, the minimal surface S has invariants $K_S^2 = 4$, $p_g = 4$, and $q = 0$. These surfaces are minimal resolutions of double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ (or $\mathbb{F}_2$) with branch divisors in the linear system $|6, 6|$ (or $6\Delta_0 + 12\Gamma$ where Δ_0 is the negative section and Γ is a fiber) (Horikawa '76).

Using this classification, we can describe all stable quintic surfaces with a $\frac{1}{4}(1,1)$ singularity by considering the image of the (-4)-curve on S under this double cover. The most general cases are seen in the figure below:

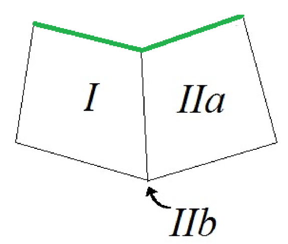


Three possibilities for the branch divisor (in green).

1. There is a family of numerical quintic surfaces of type IIb that degenerates to a stable quintic surface of type 2b (Friedman '83).
2. Following Friedman's example, one can find a family of numerical quintic surfaces of type IIa degenerating to a surface of type 2a.
3. The family of quintic surfaces
$$X_t = \{f_2^2 f_1 + t f_2 f_3 + t^2 f_5 = 0\} \subset \mathbb{C}_t \times \mathbb{P}^3,$$
where f_i is a general form of degree i, degenerates to a stable quintic surface of type 1.

The proof of the following theorem is a work in progress.

Theorem. *The locus of stable quintic surfaces with a unique $\frac{1}{4}(1,1)$ singularity is the closure of the locus of surfaces of type 1, 2a, and 2b, and is a Cartier divisor at any point in these three loci.*

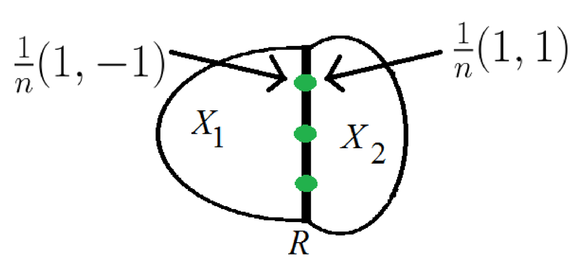


The divisor (in green) corresponding to stable quintic surfaces which have a unique $\frac{1}{4}(1,1)$ singularity.

4 Fuchsian Singularities

Recall that the moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ contains boundary divisors δ_i corresponding to irreducible curves of genera $g - i$ and i intersecting in one point. We expect analogous divisors in $\overline{\mathcal{M}}_{K^2,\chi}$ corresponding to surfaces with two components with orbifold normal crossing singularities.

Definition. An *orbifold normal crossing singularity* is locally of the form
$$(xy = 0) \subset \frac{1}{n}(1, -1, a).$$

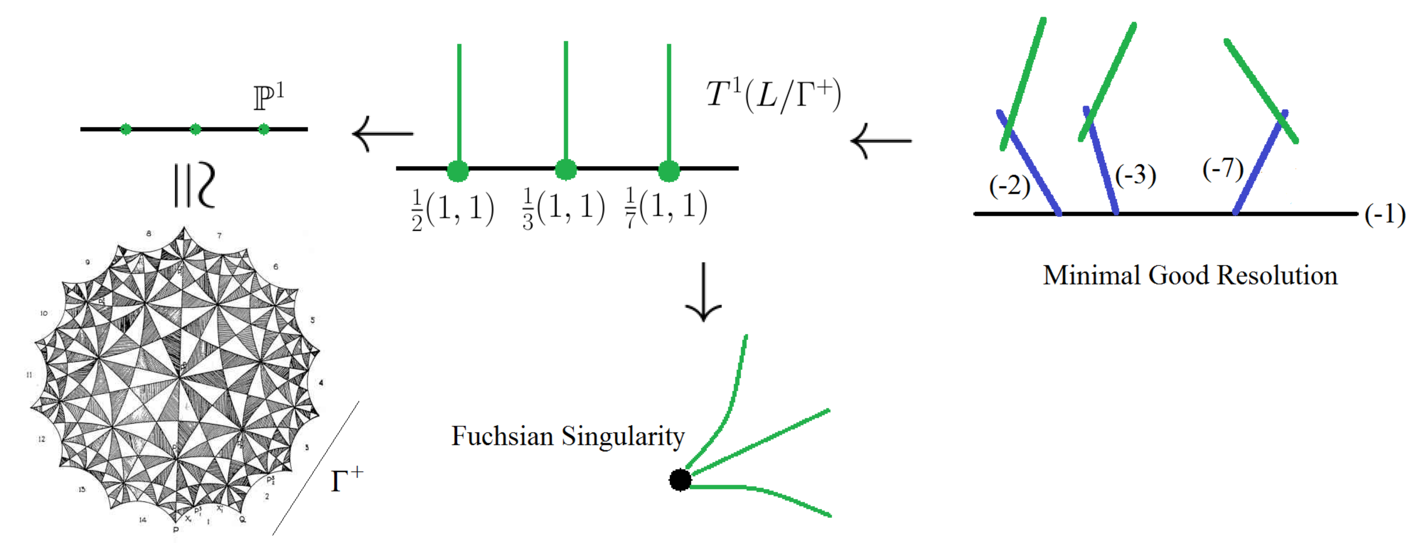


A surface with three orbifold normal crossing singularities. We see du Val singularities $\frac{1}{n}(1,-1)$ on X_1, and cyclic quotient singularities $\frac{1}{n}(1,1)$ on X_2.

An orbifold normal crossing singularity has a one-parameter smoothing given by $(xy = tf(z^n)) \subset \frac{1}{n}(1,-1,a) \times \mathbb{C}[t]$, where $f(z^n) \in \Gamma(\mathcal{O}_R(R_1|_R + R_2|_R))$. In order to obtain a divisor in $\overline{\mathcal{M}}_{K^2,\chi}$, we require that the sheaf $\mathcal{O}_R(R_1|_R + R_2|_R)$ be a sufficiently general line bundle on R of degree equal to the genus of R. For instance, if $R \cong \mathbb{P}^1$ then we require that $R_1|_R^2 + R_2|_R^2 = 0$.

The divisor δ_1 in $\overline{\mathcal{M}}_g$ corresponds to curves of genus $g-1$ with an elliptic tail. We expect comparable divisors in $\overline{\mathcal{M}}_{K^2,\chi}$ corresponding to orbifold normal crossing surfaces which are weighted blowups of surfaces containing a unique Fuchsian singularity.

To construct a Fuchsian singularity, take a tiling of the Lobachevsky plane L by a polygon with angles $\frac{\pi}{p_1}, \frac{\pi}{p_2}, \cdots, \frac{\pi}{p_r}$. Let Γ be the group generated by reflections of this tiling, and $\Gamma^+ \subset \Gamma$ the orientation-preserving subgroup. Then $L/\Gamma^+ \cong \mathbb{P}^1$. Contracting the zero-section of the tangent bundle $T^1(L/\Gamma) \to L/\Gamma$ produces the desired singularity.



The Fuchsian singularity K_{12}. The triangular tiling has angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{7}$.

Fuchsian singularities are *not* semi log canonical. However, for the 22 *exceptional Fuchsian singularities* which can be described as hypersurface singularities (Sherbak '78, Wagreich '81), we hope to obtain the stable limit of a smoothing by performing a certain weighted blowup. This results in a surface with orbifold normal crossing singularities and with one component a K3 surface containing specific du Val singularities (Dolgachev '96).

Conjecture. *Each of the 22 exceptional Fuchsian singularities corresponds to a (generically) Cartier divisor in $\overline{\mathcal{M}}_{5,5}$.*

5 Example

Let $X_t \subset \mathbb{P}^3 \times \mathbb{C}_t$ be a family of quintic surfaces locally analytically given by
$$x^2z + y^3 + z^4 + a_0t^{24} + a_1t^{15}x + a_2t^{16}y + a_3t^{18}z + a_4t^7xy + a_5t^{10}yz + a_6t^12z^2 + a_7t^4yz^2 + a_8t^6z^3 = 0$$
for some general a_i, so that X_0 has a unique exceptional Fuchsian singularity of type Q_{10}. The blowup of X_t with weights $(9, 8, 6, 1)$ has special fiber a surface with two components S_1 and S_2 meeting along a double curve R of genus 0: S_1 is a K3 surface with three singularities A_1, A_2 and A_8 along $R|_{S_1}$ and S_2 has singularities $\frac{1}{2}(1,1), \frac{1}{3}(1,1), \frac{1}{9}(1,1)$ along $R|_{S_2}$. One can check that $R|_{S_1}^2 + R|_{S_2}^2 = 0$.

We calculate the dimension of the moduli space of these surfaces. For the K3 component, we resolve the singularities to get an M-polarized K3 surface where M is a lattice of signature (1, 11). The moduli space $\mathbf{K}_M$ of these M-polarized K3 surfaces has dimension 8 (Dolgachev '96).

The minimal resolution $\tilde{S}_2$ of S_2 is the minimal good resolution of X_0. The minimal model Y of $\tilde{S}_2$ is a minimal surface with invariants $K^2 = 2, p_g = 3, q = 0$ and so is a double cover of $\mathbb{P}^2$ branched over a curve of degree 8 (Horikawa '76, Yang '84) and contains a cusp with self-intersection (-3). The moduli of such surfaces can be identified with the moduli of octic curves in $\mathbb{P}^2$ which are tangent to a cuspidal curve at 12 points. This has dimension 31. Combining with the above calculations, we see that the locus of stable quintic surfaces arising in this way is indeed 39 dimensional.