Apolarity for determinants and permanents of generic matrices

by

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1 Introduction

We show that the annihilator ideal (in the sense of the apolar pairing; i.e., Macaulay's inverse system) of the determinant and the permanent of a generic matrix, and the annihilator ideal of the Pfaffian of a generic skew symmetric matrix are generated in degree two. As a consequence we give a lower bound for the cactus rank and rank of the determinant and the permanent using a result of K. Ranestad and F. O. Schreyer.

2 Background

Let $R = \mathsf{k}[\mathsf{a_{ij}}]$ be a polynomial ring and $S = \mathsf{k}[\mathsf{d_{ij}}]$ be the ring of differential operators on R. S acts on R by contraction:

$$(d_{ij})^k \circ (a_{uv})^\ell = \begin{cases} a_{uv}^{\ell-k} & \text{if } (i,j) = (u,v), \\ 0 & \text{otherwise.} \end{cases}$$

Given $F \in R_d = \mathsf{k}[\mathsf{a_{ij}}]_\mathsf{d}$ homogeneous of degree d. A representation

$$F = l_1^d + \dots + l_r^d, \text{ with } l_i \in R_1,$$

is a Waring decomposition of length r of F.

The rank of F, $r(F)$, is defined to be the minimal r, such that F has a Waring decomposition of length r. The key object in this research is $\mathrm{Ann}(F) \subset S$ defined as

$$(\mathrm{Ann}(F))_k = \{h \in S_k | h \circ F = 0\}.$$

Let $F \in R_d$. The apolarity action of S on R, defines S as a natural coordinate ring on the projective space $\mathbf{P}(S_1)$ of 1-dimensional subspaces of S_1. A finite subscheme $\Gamma \subset \mathbf{P}(S_1)$ is apolar to F if the homogeneous ideal $I_\Gamma \subset S$ is contained in $\mathrm{Ann}(F)$ ([IK],[RS]). Define the following ranks ([IK] Def. 5.66 , [BR] and [RS])

- The cactus rank (scheme rank) $cr(F)$:
$$cr(F) = \min\{\deg\Gamma | \Gamma \subset \mathbf{P}(S_1), \dim\Gamma = 0, I_\Gamma \subset \mathrm{Ann}(F)\}.$$
- The smoothable rank $sr(F)$:
$$sr(F) = \min\{\deg\Gamma | \Gamma \subset \mathbf{P}(S_1) \text{ smoothable}, \dim\Gamma = 0, I_\Gamma \subset \mathrm{Ann}(F)\}.$$
- The rank $r(F)$:
$$r(F) = \min\{\deg\Gamma | \Gamma \subset \mathbf{P}(S_1) \text{ smooth}, \dim\Gamma = 0, I_\Gamma \subset \mathrm{Ann}(F)\}.$$
- The differential rank:
$$l_{diff}(F) = \max\{H(S/\mathrm{Ann}(F))_i\}.$$

Proposition 1. *([IK], Proposition 6.7C) The above ranks satisfy*

$$l_{diff}(F) \le cr(F) \le sr(F) \le r(F).$$

Proposition 2. ***(Ranestad-Schreyer)*** *If the ideal of $\mathrm{Ann}(F)$ is generated in degree d and $\Gamma \subset \mathbf{P}(T_1)$ is a finite apolar subscheme to F, then*

$$\deg\Gamma \ge \frac{1}{d}\deg(\mathrm{Ann}(F)),$$

where $\deg(\mathrm{Ann}(F)) = \dim(S/\mathrm{Ann}(F))$ is the length of the 0-dimensional scheme defined by $\mathrm{Ann}(F)$.

3 Main results

3.1 generic matrices

Let $A = (a_{ij})$ and $D = (d_{ij})$ be two generic matrices with entries in the polynomial ring $R = \mathsf{k}[a_{ij}]$, and in the ring of differential operators $S = \mathsf{k}[d_{ij}]$, respectively. Let $\{\mathcal{P}_A\}$, $\{\mathcal{M}_A\}$, $\{\mathcal{P}_D\}$ and $\{\mathcal{M}_D\}$ be the set of all 2×2 permanents and the set of all 2×2 minors of A and D, respectively. And let $\mathcal{P}_A$, $\mathcal{M}_A = M_2(A)$, $\mathcal{P}_D$ and $\mathcal{M}_D = M_2(D)$ be the spaces they span respectively.

Theorem 3. *Let A be a generic $n \times n$ matrix.*

1. *The apolar ideal $\mathrm{Ann}(\det(A)) \subset S$ is the ideal $(\mathcal{P}_D + \mathcal{U}_D)$, generated in degree two.*
2. *the apolar ideal $\mathrm{Ann}(\mathrm{Per}(A)) \subset S$ to $\mathrm{Per}(A) \in R$ is the ideal $(\mathcal{M}_D + \mathcal{U}_D)$, generated in degree two.*

We also have

$$H(S/\mathrm{Ann}(\det A))_k = H(S/\mathrm{Ann}(PermA))_k = \binom{n}{k}^2.$$

So we have

$$\dim_{\mathsf{k}}(S/\mathrm{Ann}(\det A)) = \sum_{k=0}^{k=n}\binom{n}{k}^2 = \binom{2n}{n}.$$

Hence we have

Theorem 4. *Let F be the determinant or permanent of a generic $n \times n$ matrix A. We have*

$$\frac{1}{2}\binom{2n}{n} \le cr(F) \le sr(F) \le r(F).$$

3.2 generic skew-symmetric matrices

Let $X^{sk} = (x_{ij})$ and $Y^{sk} = (y_{ij})$ be two generic skew-symmetric matrices with entries in the polynomial ring $R^{sk} = \mathsf{k}[x_{ij}]$, and in the ring of differential operators $S^{sk} = \mathsf{k}[y_{ij}]$. Let $Pf(X^{sk})$ denote the Pfaffian of X^{sk}.

Theorem 5. *Let X^{sk} be a generic skew-symmetric $2n \times 2n$ matrix. Then the apolar ideal $\mathrm{Ann}(Pf(X^{sk}))$ is the ideal W generated in degree 2.*

The dimension of the space of $2t \times 2t$ Pfaffians of a $2n \times 2n$ generic skew symmetric matrix $X^s k$ is $\binom{2n}{2t}$[HT]. So we have

$$\dim(S^{sk}/\mathrm{Ann}(Pf(X^{sk}))) = 2^{2n-1}.$$

So we have

$$\frac{1}{2}(2^{2n-1}) = 2^{2n-2} \le cr(Pf(X^{sk})) \le sr(Pf(X^{sk})) \le r(Pf(X^{sk})).$$

3.3 generic symmetric matrices

Let $X^s = (x_{ij})$ and $Y^s = (y_{ij})$ be two generic symmetric matrices with entries in the polynomial ring $R^s = \mathsf{k}[x_{ij}]$, and in the ring of differential operators $S^s = \mathsf{k}[y_{ij}]$. Let $Hf(X^s)$ denote the Hafnian of X^s.

Let X^s be a generic symmetric $2n \times 2n$ matrix. Then the apolar ideal $\mathrm{Ann}(Hf(X^s))$ is generated in degree 2.

$$\frac{1}{2}(2^{2n-1}) = 2^{2n-2} \le cr(Hf(X^s)) \le sr(Hf(X^s)) \le r(Hf(X^s)).$$

The dimension of the space of $t \times t$ minors of a $n \times n$ symmetric matrix is equal to the number of the doset t-minors of the $n \times n$ symmetric matrix[CON]. This is equal to the number of fillings of a semi-standard Young tableaux of shape (n,n) with the numbers $\{1, ..., n\}$, which is equal to the Narayana number $\binom{n+1}{t}\binom{n+1}{t+1}/(n+1)$. $\deg(\mathrm{Ann}(\det(X^s))$ is the Catalan number $C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$. For a generic symmetric $n \times n$ matrix X^s using Landberg-Teitler formula we have

$$r(\det(X^s)) \ge \frac{\binom{n+1}{t}\binom{n+1}{t+1}}{n+1} + \frac{(n-t-1)(n-t)}{2} + (t+1)(n-t-1) + 1.$$

$\mathrm{Ann}(\det(X^s))$ in degree 2 is the span V of certain permanents and Hafnians related to X^s. We are in the process of showing that $\mathrm{Ann}(\det(X^s))$ is also generated in degree 2.

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