The Schottky problem in genus 5

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1 Introduction

The Schottky problem is a classical problem in algebraic geometry. Let C be a curve of genus g, and $\alpha_1,\ldots,\alpha_g,\beta_1,\ldots,\beta_g$ a symplectic basis for $H^1(C,\mathbb{Z})$. Then we can choose a basis for holomorphic 1-forms $\omega_1,\ldots,\omega_g$ such that $\int_{\alpha_j}\omega_i=\delta_{i,j}$. This gives us a matrix whose entries are $\int_{\beta_j}\omega_i$ called the period matrix Ω. This matrix is well-defined up to symplectic change of basis and is an invariant of C.

Theorem (Torelli Theorem)**.** *The map associating to each curve C the orbit of its period matrix Ω is injective.*

The Schottky problem asks for conditions on which a potential period matrix is actually the period matrix of a curve, most formally, it asks for equations for the locus of Jacobians among all abelian varieties.

2 Notation

$\mathcal{M}_g$	moduli space of genus g curves
$\mathcal{A}_g$	moduli space of g-dimensional ppav's
$\mathcal{J}_g:\mathcal{M}_g\to\mathcal{A}_g$	The Torelli map, and also the image of the map
$\mathcal{RM}_g$	$\{(C,\mu)\mid C\in\mathcal{M}_g,\mu$ a nontrivial point of order 2 on $\mathcal{J}_g(C)\}$
$\mathcal{P}_g:\mathcal{RM}_g\to\mathcal{A}_{g-1}$	The Prym map $\ker^0(\mathrm{Nm})$
$\mathcal{RA}_g$	$\{(A,\mu)\mid A\in\mathcal{A}_g,\mu$ a nontrivial point of order 2 on $A\}$
$\mathcal{RJ}_g:\mathcal{RM}_g\to\mathcal{RA}_g$	the lift of $\mathcal{J}_g$ to $\mathcal{RM}_g$, and its image
$\mathcal{RC}^0$	Intermediate Jacobians of cubic 3folds with a marked even point of order 2

3 Theta Functions

We want to find equations on $\mathcal{A}_5$ for the closure of the image of $\mathcal{J}_5$. The natural coordinates to do this with live on $\mathbb{H}_g$, the Siegel upper half space of $g\times g$ symmetric matrices with positive definite imaginary part. If $\epsilon,\delta\in(\mathbb{Q}/\mathbb{Z})^g$, we have theta functions

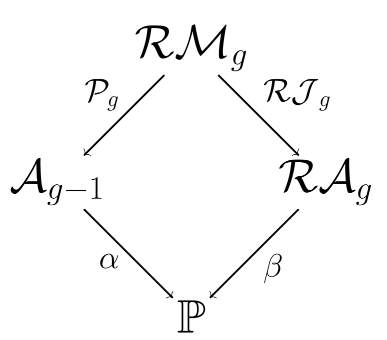
$$\theta\begin{bmatrix}\epsilon\\ \delta\end{bmatrix}(\Omega,z)=\sum_{n\in\mathbb{Z}^g}\exp\left[\pi i\left((n+\epsilon)^t\Omega(n+\epsilon)+2(n+\epsilon)^t(z+\delta)\right)\right]$$

and these can be used to construct maps:

$$\alpha_\epsilon=\theta\begin{bmatrix}\epsilon\\ 0\end{bmatrix}(2\Omega,0)\quad \epsilon\in\left(\left(\frac{1}{2}\mathbb{Z}\right)/\mathbb{Z}\right)^g$$

$$\beta_\epsilon=\theta\begin{bmatrix}\epsilon & 0\\ 0 & 1/2\end{bmatrix}(2\Omega,0)\quad \epsilon\in\left(\left(\frac{1}{2}\mathbb{Z}\right)/\mathbb{Z}\right)^{g-1}$$

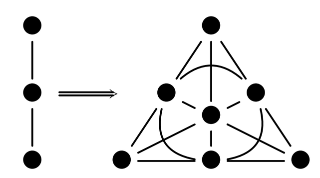
These maps α and β define maps from $\mathcal{A}_g$ and $\mathcal{RA}_g$ to a quotient of projective space $\mathbb{P}$, and Schottky and Jung [SJ09] proved that the diagram commutes.



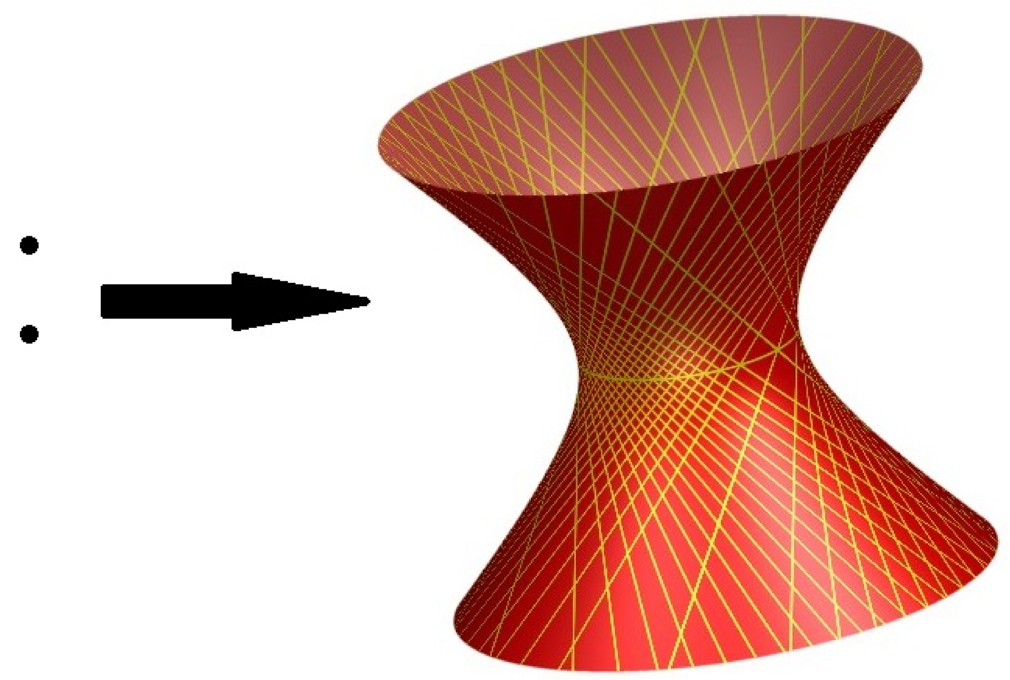
They conjectured that $\mathcal{RS}_g=\beta^{-1}(\alpha(\mathcal{A}_{g-1}))$ is precisely $\mathcal{RJ}_g$, the locus of Jacobians in $\mathcal{RA}_g$. This is not quite correct, and we work out a description of $\mathcal{RS}_5$.

4 Incidence Relations

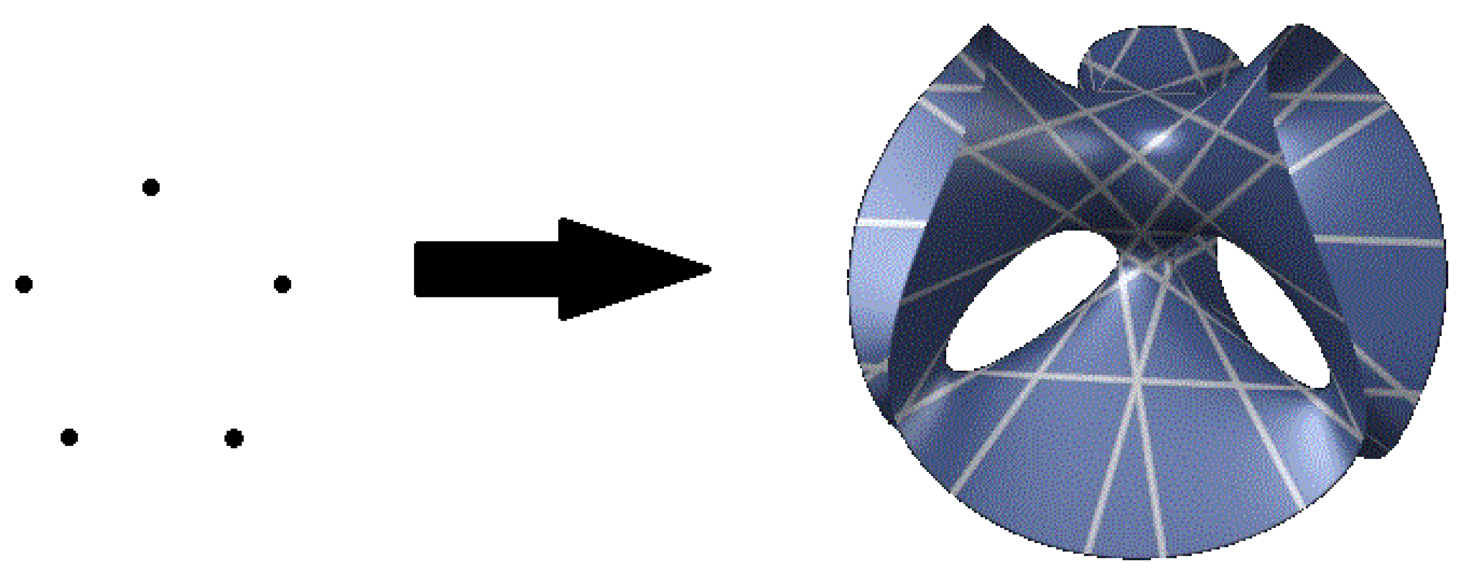
We use configurations of points and lines over finite fields to study the structure of the fibers of maps. Given a configuration V of this type, we can study configurations where the lines through each point have the configuration V. One example is that if we start with $\mathbb{P}^1(k)$ we get $\mathbb{P}^2(k)$:



Similarly, if we start with two points that are not colinear, the configuration we get is a space quadric, with each point lying on exactly two lines:



And a significant example which occurs in studying the Prym map $\mathcal{P}_6:\mathcal{RM}_6\to\mathcal{A}_5$, which uses the tetragonal construction, if we start with five points, the twenty-seven lines on the cubic surface is such a configuration:



The most significant example comes from the fact that we can obtain a natural configuration on the fibers of β_5, such that each point is on 27 lines, configured like the lines on a cubic surface, and the same construction gives us the configuration of points and lines on the quadric

$$V(x_1^2+x_1x_2+x_2^2+x_3x_4+x_5x_6+x_7x_8)\subset\mathbb{P}^7(\mathbb{F}_2)$$

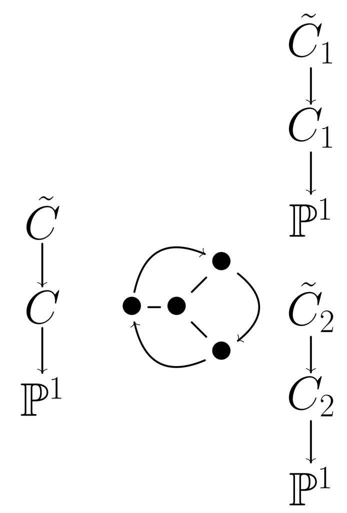
This tells us that the fibers of β_5, generically, are unions of this incidence, and so is a multiple of 119.

5 Degenerations of abelian varieties

To see that $\deg\beta=119$, degenerate to the boundary, where results from [Iza91] and [Don87] imply that there is a component where the map is generically unramified, and everything is reduced to the same behavior on the Prym map.

6 Contracted Loci

In [Don81], Donagi described the tetragonal construction, which takes a tower of a double cover of a tetragonal curve to two more such towers, and is a triality:



But also, all three double covers give the same Prym variety, and so between that and a theorem of Mumford, we can determine all of the fibers of the Prym map.

Theorem ([Mum74])**.** *Let C be a smooth non-hyperelliptic curve of genus 5. If there exists a positive dimensional family of g^1_4's on C, we must have that C is trigonal, bielliptic or a plane quintic.*

And a detailed analysis along with the above tells us that the only locus in $\mathcal{RS}_5$ that β_5 contracts is $\mathcal{RA}_1\times\mathcal{A}_4$.

7 Schottky-Jung Relations

Between the above, and the fact that the local degree of β on $\mathcal{RC}^0$ is 1, $\mathcal{RJ}_5$ is 54 and $\partial^I\mathcal{RA}_5$ is 64, we have:

Theorem.

$$\mathcal{RS}_5=\mathcal{RJ}_5\cup\mathcal{RC}^0\cup\partial^I\mathcal{RA}_5\cup\mathcal{RA}_1\times\mathcal{A}_4$$

And this theorem solves the Schottky problem in genus five, giving us that $\mathcal{S}_5^{\text{small}}=\mathcal{J}_5$.

References

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