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EQUATIONS OF RIEMANN SURFACES WITH AUTOMORPHISMS

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Introduction

Riemann surfaces (or complex algebraic curves) with automorphisms have been important objects of study for over 150 years. Group theorists have carried out computer searches and published lists of pairs (G, g) for which there exist curves of genus g with automorphism group G, but we don't know equations of all these curves; there is no published algorithm yet that, given the group, produces equations of the curve.

Here I illustrate a pseudoalgorithm that I implemented in `Magma` that produces canonical equations of all the genus 4, 5, and 6 curves satisfying $\#\operatorname{Aut}(C) > 4(g-1)$.

The main pseudoalgorithm in an example

INPUT: a finite group G, the genus g, and the ramification data $(e_1, \ldots, e_r)$ of the quotient morphism $C \to C/G$.

Step 1 (algorithmic): Compute the conjugacy classes and character table of G, and find a set of surface kernel generators *(defined below).*

Example: Let G be the following group of order 32:

$$G = \left\langle g_1, g_2, g_3, g_4, g_5 \;\middle|\; \begin{array}{l} g_1^2, g_3^2, g_4^2, g_5^2, \\ (g_1g_4)^2, (g_3g_4)^2, (g_1g_5)^2, (g_3g_5)^2, (g_4g_5)^2, \\ g_2^2g_4^{-1}, g_1g_2g_1g_4g_2^{-1}, g_1g_3g_1g_5g_3, \\ g_2^{-1}g_3g_2g_3, g_2^{-1}g_4g_2g_4, g_2^{-1}g_5g_2g_5 \end{array} \right\rangle.$$

There is a one-parameter family of curves of genus 5 whose full automorphism group is the group G. The quotient curve C/G is $\mathbb{P}^1$, and the quotient morphism $C \to C/G$ branches over 4 points of $\mathbb{P}^1$ with ramification indices $(2, 2, 2, 4)$.

We use `Magma` to compute the conjugacy classes and character table of G:

Class:	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Rep:	Id	g_5	g_4	g_4g_5	g_3	g_3g_4	g_1g_2	g_1	$g_2g_3g_5$	g_2g_3	g_2g_5	g_2	g_1g_3	$g_1g_2g_3$
Order:	1	2	2	2	2	2	2	2	4	4	4	4	4	4
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	−1	−1	−1	−1	−1	−1	1
χ_3	1	1	1	1	−1	−1	−1	−1	−1	−1	1	1	1	1
χ_4	1	1	1	1	−1	−1	−1	1	1	1	−1	−1	−1	1
χ_5	1	1	1	1	1	1	−1	−1	1	1	1	1	−1	−1
χ_6	1	1	1	1	1	1	−1	1	−1	−1	−1	−1	1	−1
χ_7	1	1	1	1	−1	−1	1	1	−1	−1	1	1	−1	−1
χ_8	1	1	1	1	−1	−1	1	−1	1	1	−1	−1	1	−1
χ_9	2	2	−2	−2	2	−2	0	0	0	0	0	0	0	0
χ_{10}	2	−2	2	−2	0	0	0	0	0	0	−2	2	0	0
χ_{11}	2	−2	2	−2	0	0	0	0	0	0	2	−2	0	0
χ_{12}	2	2	−2	−2	−2	2	0	0	0	0	0	0	0	0
χ_{13}	2	−2	−2	2	0	0	0	0	−2i	2i	0	0	0	0
χ_{14}	2	−2	−2	2	0	0	0	0	2i	−2i	0	0	0	0

Definition. A set of *surface kernel generators* for the group G and ramification data $(e_1, \ldots, e_r)$ is a sequence of elements $M_i \in G$ such that the M_i generate G, $\operatorname{Order}(M_i) = e_i$, and $M_1M_2 \cdots M_r = 1$.

We search through 4-tuples in G to find a set of surface kernel generators, and obtain $(M_1, M_2, M_3, M_4) = (g_3g_4, g_1g_4, g_1g_2g_4, g_2g_3)$.

Step 2 (algorithmic): Let $S = \operatorname{Sym} H^0(C, K)$, and let I be the canonical ideal of C. Use the Eichler Trace Formula to compute the character of the action of G on S_1 as well as on I_2 and I_3.

Eichler Trace Formula. *Suppose $g \geq 2$, and let σ be an automorphism of C of order $h \geq 2$. Write χ_m for the character of the representation of $\operatorname{Aut}(C)$ on $H^0(C, mK)$. Then*

$$\chi_m(\sigma) = \begin{cases} 1 + \displaystyle\sum_{\substack{1 \leq u < h \\ (u,h)=1}} |\operatorname{Fix}_{C,u}(\sigma)| \frac{\zeta_h^u}{1 - \zeta_h^u} & \text{if } m = 1 \\ \displaystyle\sum_{\substack{1 \leq u < h \\ (u,h)=1}} |\operatorname{Fix}_{C,u}(\sigma)| \frac{\zeta_h^{u(m\%h)}}{1 - \zeta_h^u} & \text{if } m \geq 2 \end{cases}$$

The fixed point loci $\operatorname{Fix}_{C,u}(\sigma)$ may be counted by group theoretic methods.

By Noether's Theorem, the sequence

$$0 \to I_m \to S_m \to H^0(C, mK) \to 0$$

is exact for all $m \geq 1$, and so we can compute the character of the action of G on I_2 and I_3 from knowing $\chi_1, \chi_2, \chi_3, \operatorname{Sym}^2 \chi_1$ and $\operatorname{Sym}^3 \chi_1$.

Notation: Let V be a G-module, and let $V \cong \bigoplus_{i=1}^r V_i^{\oplus m_i}$ be its decomposition into irreducible G-modules. We display the multiplicities m_i:

	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}	m_{13}	m_{14}
S_1	0	0	1	0	0	0	0	0	1	0	0	0	1	0
$H^0(C, K)$	0	0	1	0	0	0	0	0	1	0	0	0	1	0
I_2	1	1	0	0	0	1	0	0	0	0	0	0	0	0
S_2	2	2	0	0	0	2	1	0	0	1	1	1	0	1
$H^0(C, 2K)$	1	1	0	0	0	1	1	0	0	1	1	1	0	1

Step 3 (pseudoalgorithmic): Obtain matrix generators for the action of G.

At the time of this writing, it seems that there is no general algorithm that, given a finite group G and an irreducible character χ of G, produces matrix generators for a representation affording χ. However, algorithms are known in many special cases, covering most representations of degree ≤ 100.

We use `Magma` to obtain matrix representatives of the surface kernel generators under $\rho : G \to GL(5)$:

$$\rho(M_1) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & -i & 0 \end{pmatrix}, \quad \rho(M_2) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\rho(M_3) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(M_4) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & i \end{pmatrix}$$

Step 4 (algorithmic): Use the projection formula to obtain candidate quadrics (and cubics if necessary).

Let $S = \operatorname{Sym} H^0(C, K) \cong \mathbb{C}[a, b, c, d, e]$. From Step 3, we have matrix generators of the action on $S_1 = H^0(C, K)$, and hence we can compute the actions on S_2 and S_3.

We use the projection formula: the projection of S_p onto its isotypical component $S_{p,i} \cong V_i^{\oplus m_{p,i}}$ is given by

$$\pi_i = \dim V_i \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{V_i}(g)} g.$$

Then we (noncanonically) choose bases of irreducibles in $S_{p,i}$ such that G acts by the same matrix on each block.

We saw above that $I_2 \cong V_1 \oplus V_2 \oplus V_6$. We decompose the vector space of quadrics S_2 into its isotypical components:

$$S_{2,1} = \operatorname{Span}\langle a^2, b^2 + c^2 \rangle$$
$$S_{2,2} = \operatorname{Span}\langle bc, d^2 - e^2 \rangle$$
$$S_{2,6} = \operatorname{Span}\langle b^2 - c^2, de \rangle$$

Thus, we seek coefficients $\mu_1, \ldots, \mu_6$ such that

$$I_C = \langle \mu_1 a^2 + \mu_2(b^2 + c^2), \mu_3 bc + \mu_4(d^2 - e^2), \mu_5(b^2 - c^2) + \mu_6 de \rangle.$$

At this point, the algorithmic steps are done. However, we can proceed as follows.

Suppose that none of the coefficients μ_i is zero. Then we can set $\mu_1 = \mu_3 = \mu_5 = 1$:

$$I_C = \langle a^2 + \mu_2(b^2 + c^2), bc + \mu_4(d^2 - e^2), (b^2 - c^2) + \mu_6 de \rangle$$

By scaling a, we may assume that $\mu_2 = 1$, and by scaling d and e, we may assume that $\mu_4 = 1$. Thus we obtain

$$I_C = \langle a^2 + b^2 + c^2, bc + d^2 - e^2, b^2 - c^2 + \mu_6 de \rangle.$$

Now μ_6 is the only parameter remaining. However, we expected a 1-dimensional family of curves with automorphism group G, and we have obtained the desired pencil.

References

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