Partial compactification of the zero section of the universal abelian variety

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Statement of results

We investigate the zero section $0_g'$ of the partial compactification of the universal family of abelian varieties $\mathcal{X}_g' \to \mathcal{A}_g'$. We show that the class of the zero section in $H^{2g}(\mathcal{X}_g', \mathbb{Q})$ is a polynomial in three natural geometric classes of codimensions one and two. Namely, we show that there exists a formula

$$[0_g'] = \sum_{a+b+2c=g} \eta_{a,b,c} \Theta^a D^b \Delta^c,$$

where Θ is the universal theta divisor on $\mathcal{X}_g'$, D is the class of the boundary of $\mathcal{X}_g'$, and Δ is a codimension two class that is the singular locus of the boundary. We also find explicit expressions for the coefficients $\eta_{a,b,c}$.

The moduli space $\mathcal{A}_g$ and its compactifications

A *principally polarized abelian variety* $(A, [L])$ is a complex projective torus A together with the first Chern class of an ample line bundle L on A with $h^0(A, L) = 1$. A ppav is determined by its *period matrix* τ, a symmetric matrix with positive definite imaginary part.

The set of ppavs of dimension g forms a *moduli space* $\mathcal{A}_g$ of dimension $g(g+1)/2$ which admits a *universal family* $\mathcal{X}_g \to \mathcal{A}_g$ whose fiber over τ is the abelian variety A_τ. The universal family admits a *zero section* $0 : \mathcal{A}_g \to \mathcal{X}_g$ that sends τ to the identity element of A_τ.

The moduli space $\mathcal{A}_g$ is not compact. There are several ways to compactify it.

- The *Satake compactification*

$$\overline{\mathcal{A}}_g^S = \mathcal{A}_g \sqcup \mathcal{A}_{g-1} \sqcup \cdots \sqcup \mathcal{A}_1 \sqcup \mathcal{A}_0$$

  is obtained by allowing entries of the period matrix to go to infinity, and by removing the corresponding row and column. This compactification is very singular.
- The *partial compactification* in codimension one

$$\mathcal{A}_g' = \mathcal{A}_g \sqcup \mathcal{X}_{g-1}/\pm$$

  parametrizes semi-abelian varieties of torus rank one. It is obtained by allowing entries to go to infinity but retaining the row vector as a point on the corresponding rank $g-1$ abelian variety. The universal family extends to a family $\mathcal{X}_g' \to \mathcal{A}_g'$. This is the main object of our study.
- A *toroidal compactification* is obtained by instead allowing the imaginary part of the period matrix to become positive semidefinite. Of particular interest are the *perfect cone* compactification $\overline{\mathcal{A}}_g^P$, also known as the *first Voronoi*, and the *second Voronoi* compactification $\overline{\mathcal{A}}_g^{Vor}$.

Cohomology and Chow rings of $\mathcal{A}_g$ and its compactifications.

Questions. *What is the cohomology ring $H^*(\mathcal{A}_g)$? What is the Chow ring $CH^*(\mathcal{A}_g)$? What about the various compactifications?*

All of these questions are very difficult, and no general answers are known. The rings have been computed only for small genera:

- For $g = 1$ and $g = 2$ the results are classical.
- For $\mathcal{A}_3$ and $\mathcal{A}_3^S$ by Hain in 2002, for $\overline{\mathcal{A}}_3$ by van der Geer in 1998.
- For $\overline{\mathcal{A}}_4^{Vor}$ and $\overline{\mathcal{A}}_4^P$ by Hulek and Tommasi in 2011.

Approach. Instead of looking at the entire ring, only consider the *tautological subring* generated by the Chern classes λ_i of the Hodge bundle $\mathbb{E} = \pi_*(\Omega^1_{\mathcal{X}_g/\mathcal{A}_g})$. Here we have a complete answer.

Theorem (van der Geer 1999, Esnault–Viehweg 2002). The tautological subring of $CH^*_{\mathbb{Q}}(\overline{\mathcal{A}}_g)$ of a certain toroidal compactification of $\mathcal{A}_g$ has only one relation

$$(1 + \lambda_1 + \lambda_2 + \cdots + \lambda_g)(1 - \lambda_1 + \lambda_2 - \cdots + (-1)^g \lambda_g) = 1.$$

The tautological subring of $CH^*(\mathcal{A}_g)$ has an additional relation $\lambda_g = 0$, in particular λ_g is supported on the boundary of $\overline{\mathcal{A}}_g$.

Approach. Another way to study the cohomology and Chow rings of $\mathcal{A}_g$ is to explicitly construct geometric classes on $\mathcal{A}_g$ and its compactifications and compute their classes. For example, the boundary of the partial compactification $\mathcal{A}_g'$ is the universal abelian family $\mathcal{X}_{g-1}$ which has a zero section 0_{g-1}.

It turns out that these two classes are the same:

Theorem (Ekedahl, van der Geer, 2004). On the partial compactification $\mathcal{A}_g'$, the class λ_g is proportional to the zero section of $\mathcal{X}_{g-1}$:

$$\lambda_g = (-1)^g \zeta(1 - 2g)[0_{g-1}].$$

The class λ_g appears in the λ_g-*conjecture* on the intersections of λ_g with the ψ-classes on $\mathcal{M}_{g,n}$.

The λ_g-conjecture (proved by Faber and Pandharipande in 1999):

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \lambda_g = \binom{2g+n-3}{\alpha_1, \ldots, \alpha_n} \int_{\overline{\mathcal{M}}_{g,1}} \psi^{2g-2} \lambda_g = \binom{2g+n-3}{\alpha_1, \ldots, \alpha_n} \frac{2^{2g-1} - 1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}.$$

The proof of the λ_g-conjecture is hard and indirect, and it would be useful to have better understanding of the class λ_g.

Finally, the zero section of the boundary has the property that the normalization of its closure in the perfect cone compactfication $\overline{\mathcal{A}}_g^P$ is in fact the perfect cone compactification $\overline{\mathcal{A}}_{g-1}^P$ of lower rank.

The zero section of $\mathcal{X}_g$ and its partial compactification

It is thus very interesting to compute the zero section of $\mathcal{X}_g$ and its compactifications. On $\mathcal{X}_g$, we have the following result:

Theorem (Hain, 2011). The zero section 0_g of the universal abelian variety $0_g : \mathcal{X}_g \to \mathcal{A}_g$ is equal in $H^{2g}(\mathcal{X}_g, \mathbb{Q})$ to

$$[0_g] = \frac{\Theta^g}{g!},$$

where Θ is the universal theta divisor on $\mathcal{X}_g$.

We are now ready to pose our main research problem.

Main Question. What is the class of the zero section of the partial compactification $\mathcal{X}_g' \to \mathcal{A}_g'$ of the universal family of abelian varieties?

We can also ask how this class pulls back to $\mathcal{M}_{g,n}$. This leads to the following result.

Application. Fix a collection of integers $\underline{d} = (d_1, \ldots, d_n) \subset \mathbb{Z}^n$ summing to zero, and consider the *double ramification cycle*

$$D_{\underline{d}} = \left\{ (C, p_1, \ldots, p_n) | \sum d_i p_i \text{ is a principal divisor on } C \right\} \subset \mathcal{M}_{g,n}.$$

An important problem coming from symplectic field theory is to compute the closure of this class in all of $\overline{\mathcal{M}}_{g,n}$

Hain gives an answer to this problem on the space $\mathcal{M}_{g,n}^{ct}$ of curves of compact type, that is curves with compact Jacobians. Computing the extension of the zero section to the partial compactification allows us to extend the formula to the open part of the boundary divisor Δ_{irr}, that is to curves whose Jacobians have torus rank at most one.

Main result

The boundary Y of the partial compactification of the universal family $\mathcal{X}_g' \to \mathcal{A}_g'$ is a $\mathbb{P}^1$- bundle over the fiber product $\mathcal{X}_{g-1} \times_{\mathcal{A}_{g-1}} \mathcal{X}_{g-1}$, with the zero section and the infinity section identified with a shift $(\tau, z, b, 0) \sim (\tau, z + b, b, \infty)$. We obtain an explicit description of the closure of the zero section in the partial compactification.

Theorem. (Grushevsky, Z., 2012). Let $0_g'$ denote the closure of the zero section of the partial compactification $\mathcal{X}_g' \to \mathcal{A}_g'$. Then the class of $0_g'$ in $H^*(\mathcal{X}_g', \mathbb{Q})$ is equal to

$$[0_g'] = \sum_{a+b+2c=g} \eta_{a,b,c} \Theta^a D^b \Delta^c,$$

where Θ is the universal theta divisor extended to the partial compactification, D is the class of the boundary and Δ is the gluing locus on the boundary. The coefficients $\eta_{a,b,c}$ are equal to

$$\eta_{a,b,c} = \frac{(-1)^{b+c}}{2^{3b+2c} a! (2c)!} \sum_{x=0}^{b} \sum_{y=0}^{\min(x, b-x)} \frac{1}{y!(x-y)!(b-x-y)!} \frac{2^y (2c-1)!!}{(2c+2y-1)!!} (2 - 2^{2c+2x}) B_{2c+2x}.$$

Details of proof

We need to study in detail the cohomology ring of the boundary Y of $\mathcal{X}_g'$. We pass to the normalization $\widetilde{Y}$ and work with four cohomology classes: the pullbacks T_1 and T_2 of the theta divisors from the two factors, the Poincaré bundle P and the class ξ of the zero section of the $\mathbb{P}^1$-bundle.

Theorem (Erdenberger, Grushevsky, Hulek, 2008). For $g \geq 4$ the Néron–Severi group of $\mathbb{Q}$-divisors on $\mathcal{X}_{g-1} \times_{\mathcal{A}_{g-1}} \mathcal{X}_{g-1}$ modulo numerical equivalence on is generated by T_1, T_2, P and the pullback of the determinant of the Hodge bundle. The Chow ring of $\widetilde{Y}$ is generated over the Chow ring of $\mathcal{X}_{g-1} \times_{\mathcal{A}_{g-1}} \mathcal{X}_{g-1}$ by the class ξ satisfying the relation

$$CH^*(\widetilde{Y}) = CH^*(\mathcal{X}_{g-1} \times_{\mathcal{A}_{g-1}} \mathcal{X}_{g-1})/(\xi^2 - \xi P).$$

We first describe the restrictions of the four classes in the main formula to the boundary.

Proposition. The pullbacks of the zero section $0_g'$, the theta divisor Θ, the boundary divisor D and the class Δ to $\widetilde{Y}$ have the form

$$[0_g']|_{\widetilde{Y}} = \frac{\xi T_1^{g-1}}{(g-1)!}, \quad \Theta|_{\widetilde{Y}} = \xi + T_1 - \frac{P}{2}, \quad D|_{\widetilde{Y}} = -2T_2, \quad \Delta|_{\widetilde{Y}} = -4\xi T_2 - P^2 + 2PT_2.$$

To prove the polynomial formula relating $[0_g']$, Θ, D and Δ we need to understand the relations between T_1, P, T_2 and ξ. We obtain an explicit description all these relations.

Proposition. The subring of $H^*(\mathcal{X}_{g-1} \times_{\mathcal{A}_{g-1}} \mathcal{X}_{g-1})$ generated by the classes T_1, P and T_2 is the quotient of the polynomial ring $\mathbb{Q}[T_1, P, T_2]$ by the relations

$$(T_1 + NP + N^2 T_2)^g = 0, \quad N \in \mathbb{Z}.$$

To prove the main formula, we use these relations and the relation $\xi^2 = \xi P$ and directly show that both sides of the equation are the same.