Modular Forms and Special Cubic Fourfolds

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Special Cubic Fourfolds

Let X be a smooth cubic fourfold in $\mathbb{P}^5 := \mathbb{P}(V)$, and Λ be the lattice of $H^4(S,\mathbb{Z})$ w.r.t the intersection $\langle , \rangle$.

Definition 1: X is *special* if it contains a surface $S \nsim h^2$, where h is the hyperplane class on X.

Definition 2: X is special of discriminant d if the discriminant of the lattice $\langle h^2, S\rangle \subset \Lambda$ is d.

Now define $C_d \subset \mathbb{P}(\mathrm{Sym}^3 V^*) = \mathbb{P}^{55}$ to be the collection of all special cubic fourfolds of discriminant d; it is a divisor of the Hilbert scheme $\mathbb{P}^{55}$.

Eg. Let S be a projective plane, so the lattice $\langle h^2, S\rangle$ is

	h^2	S
h^2	3	1
S	1	3

of discriminant 8. Then C_8 is the collection of all smooth cubic fourfolds containing a plane.

Divisors

Λ_0	the primitive cohomology $\langle h^2\rangle^\perp \subset \Lambda$
$\mathcal{D}_{\Lambda_0}$	period domain $\{\omega \in \mathbb{P}(\Lambda_0 \otimes_{\mathbb{Z}} \mathbb{C}) \mid \omega.\omega = 0, -\omega.\bar{\omega} > 0\}$
Γ_{Λ_0}	the subgroup of $Aut(\Lambda)$ stabilizing h^2
$\mathcal{X}$	the arithmetic quotient $\mathcal{D}_{\Lambda_0}/\Gamma_{\Lambda_0}$ quasi-projective of dimension 20
$\mathcal{U}$	the Zariski open subset in $\mathbb{P}(\mathrm{Sym}^3 V^*)$, parametrizing cubic fourfolds with at worst isolated simple (A-D-E) singularities
$\mathcal{C}$	moduli space of cubic fourfolds in $\mathcal{U}$

Note there is a canonical map $\phi : \mathcal{U} \to \mathcal{C}$ since cubic fourfolds in $\mathcal{U}$ are stable with G.I.T. quotient. and therefore we have a regular map $\mathcal{P} : \mathcal{C} \to \mathcal{X}$ [5].

Now let $h^2 \in L \subset \Lambda$ be a rank 2 positive lattice; define divisors on $\mathcal{X}$ by

$$D_d = \left(\sum_{\det(L)=d} \{\omega \in D \mid \omega \perp L\} \right) / \Gamma_{\Lambda_0}$$

Hassett [1] shows that:

d	$D_d \subset \mathcal{X}$
> 6	is a nonempty divisor iff $d \equiv 0, 2 \mod 6$
2	parametrizes the L.M.H structures from *determininantal cubic fourfolds*
6	parametrizes the L.M.H structures from cubic fourfolds with one double point

Degree

▸ **Degree of C_d**

Recall the following diagram

$$\mathcal{C}_d \subset \mathcal{U} \xrightarrow{\phi} \mathcal{C} \xrightarrow{\mathcal{P}} \mathcal{X} \supset D_d, \qquad \varphi := \phi \circ \mathcal{P}$$

Now, take $\pi : \mathfrak{D} \to \mathbb{P}^1$ to be a Lefschetz pencil of cubic hypersurfaces of $\mathbb{P}(V)$. It yields a natural morphism ι_π

$$\begin{array}{c} \mathfrak{D} \\ \downarrow \pi \\ \mathbb{P}^1 \xrightarrow{\iota_\pi} \mathcal{U} \xrightarrow{\varphi} \mathcal{X} \end{array}$$

Then $\deg(C_d) := \int_{\mathbb{P}^1} \iota_\pi^*[C_d] = \int_{\mathbb{P}^1} \iota_\pi^* \varphi^*[D_d]$. Geometrically, the degree could be interpreted as the number of cubic fourfolds containing a particular surface S in a Lefschetz pencil.

▸ **Enumerative computation of the degrees**

d	$\deg(C_d)$
2	0 (no *determinantal cubic fourfolds* on each fiber $\mathfrak{D}_t$)
6	192 (the top chern class of $J^1(\mathcal{O}_{\mathbb{P}(V)}(3))$)
8	3402 (the top Chern class of $\mathrm{Sym}^3 S^*$) where S is the universal subbundle of $Gr(3,V)$)
12	??

Note that C_{12} is the collection of cubic fourfolds containing a cubic scroll. It gets much harder to compute $\deg(\mathcal{C}_d)$ using enumerative geometry as d gets larger.

Heegner Divisors

For the lattice $-\Lambda_0 =: M$, an even lattice of signature $(2,20)$, we could define $\mathcal{X}_M := \mathcal{D}_M/\Gamma_M$, where $\Gamma_M := \{g \in \mathrm{Aut}(M) \mid g|_{M^\vee/M} = id\}$. Let γ_i denote a basis of $M^\vee/M$, where $\langle \gamma_i, \gamma_i\rangle = -\frac{i^2}{3} \mod \mathbb{Z}$, and v_i the corresponding basis in $\mathbb{C}[M^\vee/M]$.

Given $\gamma_i \in M^\vee/M$, $n \in \mathbb{Q}^{<0}$, $v \in M^\vee$, we have Heegner Divisors [2][4]

$$y_{n,\gamma_i} = \left(\sum_{\frac{1}{2}\langle v,v\rangle = n,\ v \equiv \gamma_i \bmod M} v^\perp \right) / \Gamma_M \quad \subset \mathcal{X}_M$$

Modular Forms

▸ **Vector-valued generating series of Heegner divisors**

Set $q = e^{2\pi i \tau}$ consider the vector-valued generating series

$$\vec{\Phi}(q) = \sum_{n \in \mathbb{Q} \geq 0} \sum_{\gamma_i \in M^\vee/M} y_{-n,\gamma_i} q^n v_\gamma \in Pic(\mathcal{X}_M)[[q^{1/N}]] \otimes_{\mathbb{Z}} \mathbb{C}[M^\vee/M] \quad .$$

We have the following result:

Theorem [3][4] The generating series $\vec{\Phi}(q)$ is an element of $Pic(\mathcal{X}_M) \otimes_{\mathbb{Z}} \mathrm{Mod}(Mp_2(\mathbb{Z}), 11, \rho_M^*)$, where $\mathrm{Mod}(Mp_2(\mathbb{Z}), 11, \rho_M^*)$ is the space of vector-valued modular forms of weight 11 and type ρ_M^*.

▸ **Vector-valued generating series of $\deg C_d$**

1) There is a $1-1$ correspondence between D_d and $y_{n,\gamma}$ by setting $n = -\frac{d}{6}$, $\gamma \equiv \frac{d}{2}\gamma_1 \mod M$

2) As a result, for any linear function $\lambda : Pic(\mathcal{X}_M) \otimes \mathbb{C} \to \mathbb{C}$, the corresponding linear contraction $\lambda(\vec{\Phi})$ is a vector-valued modular form in $\mathrm{Mod}(Mp_2(\mathbb{Z}), 11, \rho_M^*)$. Thus, we could define generating series of $\deg C_d$.

Results

Theorem 1 The generating series $\vec{\Psi}(q) := v_0 + \sum_{i=0}^{2} \sum_{d \equiv i^2 \bmod 3}^{\infty} \deg(\mathcal{C}_d) q^{\frac{d}{6}} v_i$ is in $\mathrm{Mod}(Mp_2(\mathbb{Z}), 11, \rho_M^*)$.

$$\vec{\Psi}(q) = -\vec{F_0}(q) - \frac{3}{4}\vec{F_1}(q) = (-2 + 192q + 196272q^2 + \ldots)v_0 + (3402q^{4/3} + 917568q^{7/3} + \ldots)(v_1 + v_2).$$

where $\vec{F_0}(q)$ and $\vec{F_1}(q)$ are a basis of $\mathrm{Mod}(Mp_2(\mathbb{Z}), 11, \rho_M^*)$.

Theorem 2 The generating series $\Theta(q) = -2 + \sum_{d>2}^{\infty} deg(C_d) q^{\frac{d}{6}}$ is a modular form of weight 11, level 3.

$$\begin{aligned} \Theta(q) = & -\alpha^{11} + 162\alpha^8\beta + 91854\alpha^5\beta^2 + 2204496\alpha^2\beta^3 - \alpha^{11}(q^{\frac{1}{3}}) + 66\alpha^8(q^{\frac{1}{3}})\beta(q^{\frac{1}{3}}) - 1386\alpha^5(q^{\frac{1}{3}})\beta^2(q^{\frac{1}{3}}) \\ & + 9072\alpha^2(q^{\frac{1}{3}})\beta^3(q^{\frac{1}{3}}) \\ = & -2 + 192q + 3402q^{\frac{4}{3}} + 196272q^2 + 915678q^{\frac{7}{3}} + \ldots \end{aligned}$$

where $\alpha(q) = 1 + 6\sum_{n\geq 1} q^n \sum_{d|n} \left(\frac{d}{3}\right)$, $\beta(q) = \sum_{n \geq 1} q^n \sum_{d|n} (n/d)^2 \left(\frac{d}{3}\right)$, and $\left(\frac{d}{3}\right)$ is the Legendre symbol.

Theorem 3 $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{C})$ has rank one.

Remarks:

▸ In **Theorem 1**, the existence of $\vec{\Psi}(q)$ is obtained from the 1-1 relation between $y_{n,\gamma}$ and C_d. By finding two basis elements $\vec{F_0}(q)$ and $\vec{F_1}(q)$ of $\mathrm{Mod}(Mp_2(\mathbb{Z}), 11, \rho_M^*)$, we write out $\vec{\Psi}(q)$ explicitly.

▸ $\mathcal{C}$ is an open subset of $\mathcal{X}$ via the open immersion $\mathcal{P}$ and the boundary $\mathcal{X}\backslash\mathcal{C}$ is D_2 [5]. Then a direct corollary of **Theorem 3** is that $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{X})$ has rank two and is spanned by $y_{0,0}$ and $D_2 = y_{-\frac{1}{3},\gamma_1}$.

References

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