POLYTOPES, TORIC VARIETIES, AND IDEALS

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Introduction

This summer we explored the connections between geometric objects such as polytopes and projective toric varieties and algebraic objects such as monomials and ideals. We followed a path which started at special types of polytopes called selfatopes. We associated monomials to them and then a mapping. This mapping brought us to a projective toric variety which we then associated with an ideal of a polynomial ring.

In section 1 we introduce basic definitions concerning convexity, polytopes, dimension, and selfatopes, which will be needed throughout the paper. Then in section 2 we define products and prisms of polytopes and prove that these operations preserve the characteristics of selfatopes. In section 3 we define the mapping, toric variety, and ideal associated with a general lattice polytope. We then prove several propostions about the toric varieties and ideals of the standard lattice n-simplices, pyramids, and prisms of polytopes. Finally, we end with a conjecture about the ideal of the toric variety of the product of two polytopes.

Although the results in this paper may not be new to experts, we give explicit proofs which may not appear in the literature in this form.

1. Basic Definitions

First we define some fundamental terms that will be necessary in understanding this paper.

Definition 1.1. A set C $\subset \mathbb{R}^n$ is *convex* if for every $p, q \in C$, $tp + (1-t)q \subset C$, $0 \leq t \leq 1$. In other words C is convex if for every pair of elements in C the line segment between them is also contained in C.

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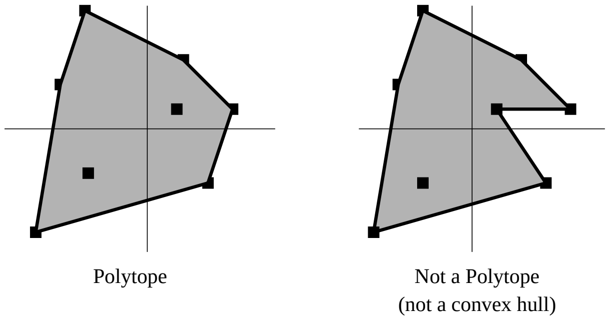
Definition 1.2. Let $A \subset \mathbb{R}^n$. The *convex hull* of A, $\text{conv}(A)$, is the intersection of all convex sets containing A. It is the smallest convex set containing A.

Definition 1.3. A *polytope* is the convex hull of a finite set of points.

Example 1.4. Here is an example of a polytope in $\mathbb{R}^2$ and a non-example which fails to be a convex-hull.



The following definitions are essential in understanding the dimension and faces of a polytope P.

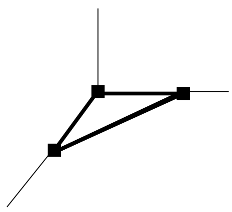
Definition 1.5. A *hyperplane* in $\mathbb{R}^n$ is the set of all solutions to a linear equation $\lambda_1 x_1 + \ldots + \lambda_n x_n = 0$ where not all $\lambda_i = 0$.

Definition 1.6. An *affine hyperplane* in $\mathbb{R}^n$ is the set of all solutions to a linear equation $\lambda_1 x_1 + \ldots + \lambda_n x_n = a$ where not all $\lambda_i = 0$ and $a \in \mathbb{R}$.

Definition 1.7. Let A be a subset of $\mathbb{R}^n$. The *affine hull* of A, $\text{aff}(A)$, is the intersection of all affine hyperplanes containing A.

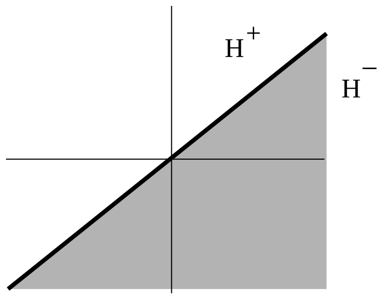
Definition 1.8. The *dimension* of a polytope P is the dimension of its affine hull. A polytope is *full-dimensional* in $\mathbb{R}^n$ if the dimension of P is n.

Example 1.9. Here is a triangle in $\mathbb{R}^3$ which lies entirely in the xy-plane. Thus its affine hull is the xy-plane and its dimension is 2. This triangle is not full-dimensional in $\mathbb{R}^3$.



Definition 1.10. If H is the hyperplane $a \cdot w = \alpha$, where $\alpha \in \mathbb{R}$, then
$H^+ = \{w \in \mathbb{R}^n | a \cdot w \geq \alpha\}$
$H^- = \{w \in \mathbb{R}^n | a \cdot w \leq \alpha\}$
are the *positive* and *negative halfspaces* determined by H.

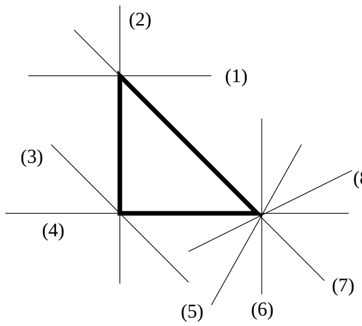
Example 1.11. Let H be the hyperplane $(-1, 1) \cdot (x, y) = 0$. Here are H^+ and H^-.



Definition 1.12. A hyperplane H is a *supporting hyperplane* of a polytope P if
(1) $H \cap P \neq \emptyset$
(2) P lies in H^+ or H^-.

Definition 1.13. If H is a supporting hyperplane of P, then $H \cap P$ is a *face* of P. 0-dimensional faces are *vertices*, 1-dimensional faces are *edges* and the $(n-1)$-dimensional faces of an n-dimensional polytope are *facets.*

Example 1.14. Here are supporting hyperplanes of the polytope P = conv$\{(0, 0), (0, 1), (1, 0)\}$.



(1) y = 1
(2) x = 0
(3) x + y = 0
(4) y = 0
(5) y - x = -1
(6) x = 1
(7) x + y = 1
(8) $\frac{-1}{2}$ x + y = $\frac{-1}{2}$

Note that a vertex has (infinitely) many supporting hyperplanes.

Definition 1.15. Let P be a polytope. Define *vert(P)*=$\{v \mid v$ is a vertex of P$\}$.

The following definitions concern the specific characteristics of the polytopes discussed in section 2 of this paper.

Definition 1.16. A *lattice point* is an element of $\mathbb{Z}^n$.

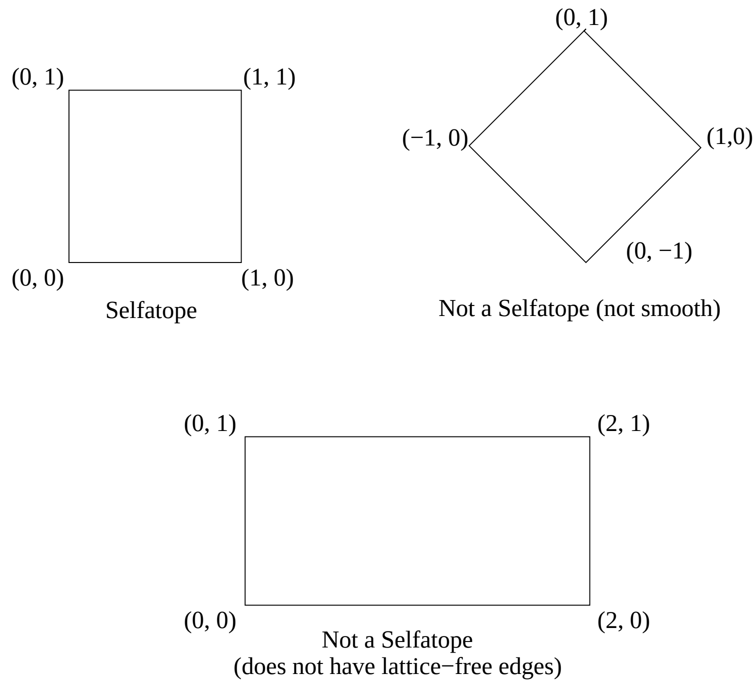
Definition 1.17. A polytope whose vertices are lattice points is a *lattice polytope*. A polytope with no lattice points on its edges except for its vertices has *lattice-free edges*.

Definition 1.18. A polytope P $\subseteq \mathbb{R}^n$ is *smooth* if for all $v \in \text{vert}(P)$, the vectors $\{w_1 - v, ..., w_k - v\}$ form part of a $\mathbb{Z}-$ basis for $\mathbb{Z}^n$, where $w_1, ..., w_k$ are the nearest lattice points to v along the edges incident to v. In other words if P is full-dimensional in $\mathbb{R}^n$ then P is smooth if for all v,

$$\det \begin{pmatrix} | & & | \\ w_1 - v & \dots & w_k - v \\ | & & | \end{pmatrix} = \pm 1.$$

Definition 1.19. A *selfatope* is a smooth lattice polytope with lattice-free edges.

Example 1.20. Here is an example of a selfatope and two non-examples which fail to be a selfatopes for different reasons.



Selfatope

Not a Selfatope (not smooth)

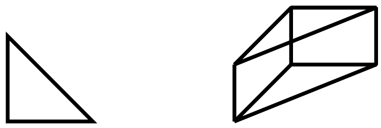
Not a Selfatope
(does not have lattice−free edges)

2. Making New Selfatopes: Prisms and Products

Once we have some basic examples of selfatopes we would like to form new selfatopes out of these examples. One way to do this is to prism a selfatope or take a product of two selfatopes. In this section we show that these constructions preserve the properties of a selfatope.

Definition 2.1. The *product* of two polytopes $P \subseteq \mathbb{R}^m$ and $Q \subseteq \mathbb{R}^n$ is $P \times Q = \{(p,q) \in \mathbb{R}^{m+n} \mid p \in P, q \in Q\}$. The product P×[0,1] is called the *prism of P*.

Example 2.2.



P prism of P

The following two propositions are well known results concerning the product of two polytopes:

Proposition 2.3. *If $P \subseteq \mathbb{R}^m$ and $Q \subseteq \mathbb{R}^n$ are polytopes, then $P \times Q$ is a polytope.*

Proposition 2.4. *The k-dimensional faces of $P \times Q$ are*

$$\{F \times H \mid F \in \text{ face }_i(P) \text{ and } H \in \text{ face }_j(Q) \text{ where } i + j = k\},$$

where $\text{face }_i(P) = \{i - \text{dimensional faces of } P\}$.

Corollary 2.5. *Let $P \subseteq \mathbb{R}^m$ and $Q \subseteq \mathbb{R}^n$ be polytopes. Vertices of $P \times Q$, are products of the vertices of P and Q.*

Theorem 2.6. *If $P \subseteq \mathbb{R}^m$ and $Q \subseteq \mathbb{R}^n$ are selfatopes, then $P \times Q$ is a selfatope.*

Proof. Let P be full-dimensional in $\mathbb{R}^m$ and let Q be full-dimensional in $\mathbb{R}^n$. We need to show that P×Q (1) is a lattice polytope, (2) has lattice-free edges, and (3) is smooth.

(1) Since P and Q are lattice polytopes P×Q is a lattice polytope, by Corollary 2.5.

(2) Let l be an edge in P×Q. So dim(l)=1. Therefore l is the product of a 0-dimensional face of P and a 1-dimensional face of Q or vice-versa by Proposition 2.4.

Let $d \in l$, $d =$ (p, point on edge in Q) or $d =$ (point on edge in P, q) where $p \in$ vert(P) and $q \in$ vert(Q). Since the edges of P and Q are lattice-free, d will not be a lattice point unless d is a vertex. So P×Q has lattice-free edges.

(3) Let $v \in$ vert(P×Q). So $v = (p,q)$ where $p \in$ vert(P) and $q \in$ vert(Q). Since P is full-dimensional in $\mathbb{R}^m$ and P is smooth each vertex has m incident edges. So the edges incident to p are $\{pp_1, ..., pp_m\}$ where $p_i \in$ vert(P), $1 \leq i \leq m$ and pp_i is the edge between p and

p_i. Similarly edges incident to q are $\{qq_1, ..., qq_n\}$ where $q_k \in$ vert(Q), $1 \le k \le n$ and qq_k is the edge between q and q_k.

So the edges incident to v are $\{p\} \times qq_k$ where $1 \le k \le n$ and $pp_i \times \{q\}$ where $1 \le i \le m$. Since these edges are lattice-free, the closest lattice points to v along these edges are the vertices $(p,\ q_1)$,...,(p,q_n), $(p_1,\ q)$, ...,$(p_m,\ q)$. In calculating smoothness we have the difference vectors $(\bar{0},\ q_1 - q)$,...,$(\bar{0},\ q_n - q)$, $(p_1 - p, \hat{0})$,...,$(p_m - p, \hat{0})$, where $\bar{0}$ is the zero-vector in $\mathbb{R}^m$ and $\hat{0}$ is the zero-vector in $\mathbb{R}^n$.

$$\text{Let A} = \begin{pmatrix} | & & | \\ p_1 - p & \dots & p_m - p \\ | & & | \end{pmatrix} \text{and B} = \begin{pmatrix} | & & | \\ q_1 - q & \dots & q_n - q \\ | & & | \end{pmatrix}.$$

Since P and Q are smooth detA = $\pm$ 1 and detB = $\pm$ 1. Thus

$$\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \pm 1.$$

Therefore P×Q is smooth. □

3. Ideals and Toric Varieties of Pyramids and Products

In this section we relate polytopes to geometric objects in projective space called varieties and then to ideals of polynomial rings. For more details concerning this material, refer to [3]. Furthermore we explore how making pyramids and products of polytopes affects these ideals. We begin with some general definitions:

Definition 3.1. Let K be a field. Let $\underline{t} = (t_1, t_2, ..., t_n) \in (K^*)^n$ and let $\underline{a} = (a_1, a_2, ..., a_n) \in \mathbb{Z}^n$. Define $\underline{t}^{\underline{a}} = t_1^{a_1} t_2^{a_2} ... t_n^{a_n}$.

Example 3.2. Let $\underline{t} = (t_1, t_2) \in (\mathbb{C}^*)^2$. So $(t_1, t_2)^{(7,4)} = t_1^7 t_2^4$.

Definition 3.3. *Affine n-space over a field K* is $\mathbb{A}_K^n = K^n$.

Definition 3.4. An *affine variety* $\subseteq \mathbb{A}_{\mathbb{C}}^n$ is the solution set of a finite number of polynomial equations.

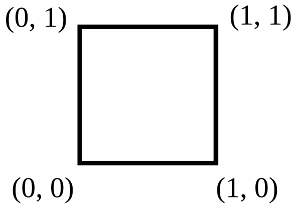
Now we relate these ideas specifically to polytopes.

Definition 3.5. Let P be a lattice polytope $\subset \mathbb{R}^n$ with $P \cap \mathbb{Z}^n = \{\underline{a}_0, ..., \underline{a}_m\}$ where $\underline{a}_i = (a_{i1}, ..., a_{in})$. Define $\phi_P : (\mathbb{C}^*)^n \longrightarrow \mathbb{P}^m$, where $\phi_P(\underline{t}) = [\underline{t}^{\underline{a}_0} : ... : \underline{t}^{\underline{a}_m}]$.

Definition 3.6. Let P be a n-dimensional lattice polytope $\subseteq \mathbb{R}^n$ with $P \cap \mathbb{Z}^n = \{\underline{a}_0, ..., \underline{a}_m\}$. The closure of the image of ϕ_P in $\mathbb{P}_{\mathbb{C}}^m$ is the *projective toric variety* X_P.

Definition 3.7. The *ideal of* X_P is $I(X_P)= \{f \in \mathbb{C}[x_0, ...x_m] \mid f(\underline{a}) = 0, \forall \underline{a} \in X_P\}$.

Example 3.8. Let $P = \text{conv}\{(0,0),(1,0),(0,1),(1,1)\}$.
$P \cap \mathbb{Z}^2 = \{(0,0),(1,0),(0,1),(1,1)\}$



So $\phi_P : (\mathbb{C}^*)^2 \longrightarrow \mathbb{P}^3$
Let $\underline{t} = (t_1, t_2) \in (\mathbb{C}^*)^2$. We have that $\phi_P(\underline{t}) = [1 : t_1 : t_2 : t_1t_2]$.

$\text{Im}(\phi_P) = \{[\lambda : \lambda t_1 : \lambda t_2 : \lambda t_1 t_2] \mid \lambda \neq 0\}$, as we account for homogeneous coordinates.

So $X_P = \overline{Im(\phi_P)}$.
Using Macaulay2 [4], $I(X_P) = \langle x_1x_2 - x_0x_3\rangle$.

Definition 3.9. The *variety of an ideal* $I \subseteq \mathbb{C}[x_0, ..., x_m]$ is $V(I) = \{\underline{a} \in \mathbb{A}^n_{\mathbb{C}} \mid f(\underline{a}) = 0, \forall f \in I\}$. $V(I)$ is closed.

Proposition 3.10. *The polynomial* $f \in \mathbb{C}[x_0, ..., x_m]$ *is zero when evaluated at every point of* X_P *if an only if* f *is zero when evaluated at every point of the image of* ϕ_P.

Proof. ($\Rightarrow$) This implication follows from $\text{Im}\,\phi_P \subseteq X_P$.
($\Leftarrow$) Let f vanish on $\text{Im}\,\phi_P$. This implies that $\text{Im}\,\phi_P \subseteq V(\langle f\rangle)$. Since $V(\langle f\rangle)$ is closed and $X_P = \overline{\text{Im}\,\phi_P}$, we have that $X_P \subseteq V(\langle f\rangle)$. Therefore f vanishes on X_P. □

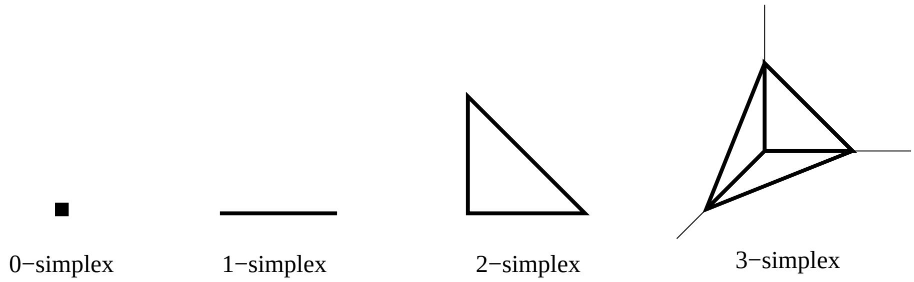
Corollary 3.11. *Let* P *be a* n*-dimensional lattice polytope* $\subseteq \mathbb{R}^n$ *with* $P \cap \mathbb{Z}^n = \{\underline{a}_0, ..., \underline{a}_m\}$.

$I(X_P) = \{f \in \mathbb{C}[x_0, ..., x_m] \mid f([\lambda \underline{t}^{\underline{a}_0} : ... : \lambda \underline{t}^{\underline{a}_m}]) = 0$ *where* $\lambda \neq 0,\ \underline{t} \in (\mathbb{C}^*)^n\}$.

Let's first look at the ideals and toric varieties of one of the most basic families of polytopes, the standard n-simplices.

Definition 3.12. A *standard lattice* n*-simplex*, $\Delta_n \subset \mathbb{R}^n$, is $\text{conv}\{\bar{0}, e_1, ..., e_n\}$ where $\bar{0}$ is the zero vector in $\mathbb{R}^n$ and $e_1, ..., e_n$ are the standard basis vectors of $\mathbb{R}^n$.

Example 3.13.



Proposition 3.14. *If P is a standard lattice n-simplex, then $I(X_P)$=$\langle 0\rangle$.*

Proof. Since $P = \text{conv}\{\bar{0}, e_1, ..., e_n\} \subset \mathbb{R}^n$, $P \cap \mathbb{Z}^n = \{\bar{0}, e_1, ..., e_n\}$. So $\phi_P(\underline{t}) = [1{:}t_1 : ... : t_n]$ where $\underline{t} = (t_1, ..., t_n)$.

$$\text{Thus } \operatorname{Im}\phi_P = \{[\lambda : \lambda t_1 : ... : \lambda t_n] \mid \lambda \neq 0,\ \underline{t} \in (\mathbb{C}^*)^n\}.$$

Suppose $I(X_P)\neq \langle 0\rangle$. So there exists an $f \in \mathbb{C}[x_0, ..., x_n]$ such that $f(\underline{x})$= 0 for all $\underline{x} \in X_P$ and f is not the zero polynomial. This implies $f(\underline{x})$= 0 for all $\underline{x} \in \operatorname{Im}\phi_P \subseteq X_P$. Hence $f([\lambda : \lambda t_1 : ... : \lambda t_n]) = 0$ $\forall\lambda \neq 0$ and $\underline{t} \in (\mathbb{C}^*)^n$.

Since f is a polynomial, $f = \sum_{i=0}^{N} \alpha_i(\underline{x}^{\underline{a}_i})$ where $\alpha_i \in \mathbb{C}$ and $\underline{a}_i \in \mathbb{N}^{n+1}$, $\underline{a}_i \neq \underline{a}_j$ $\forall i, j$.

$$\text{So } f([\lambda : \lambda t_1 : ... : \lambda t_n]) = \sum_{i=0}^{N} \alpha_i([\lambda : \lambda t_1 : ... : \lambda t_n])^{\underline{a}_i} = 0.$$

Since f is not the zero polynomial there exists i such that $\alpha_i \neq 0$.

So $f([\lambda : \lambda t_1 : ... : \lambda t_n]) = \alpha_i\lambda^k t_1^{a_{i,1}} t_2^{a_{i,2}} ... t_n{}^{a_{i,n}} + ... = 0$, where $k = a_{i,0}a_{i,1}...a_{i,n}$.

However since each term of f is a distinct monomial, as the a_j's are distinct for all j, this i^{th} term cannot be cancelled. Thus we have a contradiction and so f must be the zero polynomial and $I(X_P)$=$\langle 0\rangle$. □

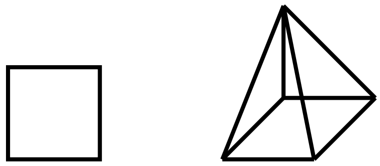
Corollary 3.15. *If P is standard lattice n-simplex, then $X_P = \mathbb{P}^n_{\mathbb{C}}$.*

Proof. Let P be an n-simplex. By Proposition 3.14, $I(X_P)$= $\langle 0\rangle$. Since X_P is a variety it is the set of points $\{\underline{x} \in \mathbb{P}_{\mathbb{C}}{}^n \mid f(\underline{x}) = 0, \forall f \in I(X_P)\}$. Therefore $X_P = \mathbb{P}^n_{\mathbb{C}}$, since every element of $\mathbb{P}^n_{\mathbb{C}}$ is a solution to the zero polynomial. □

Now that we know the varieties and ideals of varieties of n-simplices we move on to slightly more complicated polytopes. Given a lattice polytope we can form new lattice polytopes from it by constructing pyramids, prisms, and products.

Definition 3.16. Let P be a d-dimensional polytope $\subset \mathbb{R}^n$, $n > d$. The *pyramid of P*, pyr(P), is conv(P,p) where p $\notin$ aff(P). A pyramid over P has dimension $d+1$.

Example 3.17.



P pyr(P)

Proposition 3.18. *Let $P \subseteq \mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^{n+1}$ be a lattice polytope. Let pyr(P)$\subset \mathbb{R}^{n+1}$ equal* $\text{conv}(P, \underline{p})$ *where $\underline{p} = (p_1, ..., p_n, 1)$. $I(X_P)$ and $I(X_{pyr(P)})$ have the same generators.*

Proof. Let $I(X_P)$=$\langle g_1, ..., g_s\rangle$ where $g_i \in \mathbb{C}[x_0, ..., x_m] \subset \mathbb{C}[x_0, ..., x_{m+1}]$. We need to show (1) $g_i \in I(X_{pyr(P)})$ $\forall i \in \{1, ...s\}$ and (2) $\forall f \in I(X_{pyr(P)})$, $f = \beta_1 g_1 + ... + \beta_s g_s$ where $\beta_i \in \mathbb{C}[x_0, ..., x_{m+1}]$, $1 \leq i \leq s$.

(1) Let $P \cap \mathbb{Z}^n = \{\underline{a}_0, ..., \underline{a}_m\}$, so

$$\phi_P(\underline{t}) = [\underline{t}^{\underline{a}_0} : ... : \underline{t}^{\underline{a}_m}] \text{ where } \underline{t} = (t_1, ..., t_n) \in (\mathbb{C}^*)^n.$$

Thus Im ϕ_P=$\{[\lambda \underline{t}^{\underline{a}_0} : ... : \lambda \underline{t}^{\underline{a}_m}] \mid \lambda \neq 0\ \underline{t} \in (\mathbb{C}^*)^n\}$.
So $g_i([\lambda \underline{t}^{\underline{a}_0} : ... : \lambda \underline{t}^{\underline{a}_m}]) = 0$, $\forall i \in \{1, .., s\}$.

Since $p_{n+1} = 1$, the only lattice point in $pyr(P)$ that is not in P is $\underline{p}$ itself. So $pyr(P) \cap \mathbb{Z}^{n+1} = \{(\underline{a}_0, 0), ..., (\underline{a}_m, 0), (p_1, ..., p_n, 1)\}$. Let $\underline{t'} = (t_1, ..., t_n, t_{n+1}) \in (\mathbb{C}^*)^{n+1}$,

$$\phi_{pyr(P)}(\underline{t'}) = [\underline{t'}^{(\underline{a}_0,0)} : ... : \underline{t'}^{(\underline{a}_m,0)} : \underline{t'}^{\underline{p}}] = [\underline{t}^{\underline{a}_0} : ... : \underline{t}^{\underline{a}_m} : \underline{t'}^{\underline{p}}].$$

Thus Im $\phi_{pyr(P)} = \{[\lambda \underline{t}^{\underline{a}_0} : ... : \lambda \underline{t}^{\underline{a}_m} : \lambda \underline{t'}^{\underline{p}}] \mid \lambda \neq 0,\ \underline{t'} \in (\mathbb{C}^*)^{n+1}\}$.

Note that $\forall i \in \{1, ..., s\}$, $g_i([\lambda \underline{t}^{\underline{a}_0} : ... : \lambda \underline{t}^{\underline{a}_m} : \lambda \underline{t'}^{\underline{p}}]) = 0$, since g_i has no $(m+1)^{st}$ term. Therefore by Corollary 3.11, $g_i \in I(X_{pyr(P)})$, $\forall i$. This implies that

$$\langle g_1, ..., g_s\rangle \mathbb{C}[x_0, ..., x_{m+1}] \subseteq I(X_{pyr(P)}),$$

where $\langle g_1, ..., g_s\rangle \mathbb{C}[x_0, ..., x_{m+1}] = \{g_1 q_1 + g_2 q_2 + ... + g_s q_s \mid q_k \in \mathbb{C}[x_0, ..., x_{m+1}],\ 1 \leq k \leq s\}$.

(2) Let $f \in I(X_{pyr(P)})$. So $f([\lambda \underline{t}^{\underline{a}_0} : ... : \lambda \underline{t}^{\underline{a}_m} : \lambda \underline{t'}^{\underline{p}}]) = 0$, $\forall \lambda \neq 0$. Since $f \in \mathbb{C}[x_0, ..., x_{m+1}]$, $f(\underline{x}) = \sum_{j=0}^{N} \alpha_j (\underline{x}^{\underline{b}_j}) = 0$ where $\underline{b}_j \in \mathbb{N}^{m+2}$ and $\alpha_j \in \mathbb{C}$.

Now group terms of f by the exponents of x_{m+1}. So

$$f(\underline{x}) = h_1(x_0, ..., x_m) x_{m+1}^{k_1} + ... + h_\gamma(x_0, ..., x_m) x_{m+1}^{k_\gamma},$$

where $\gamma \in \mathbb{N}$, $k_w \in \mathbb{N}$, $k_w \neq k_v\ \forall w, v$, and $h_w \in \mathbb{C}[x_0, ..., x_m]$, $\forall w \in \{1, ..., \gamma\}$.
Thus $f([\lambda \underline{t}^{\underline{a}_0} : ... : \lambda \underline{t}^{\underline{a}_m} : \lambda \underline{t'}^{\underline{p}}]) =$

$$h_1([\lambda \underline{t}^{\underline{a}_0} : ... : \lambda \underline{t}^{\underline{a}_m}])(\lambda \underline{t'}^{\underline{p}})^{k_1} + ... + h_\gamma([\lambda \underline{t}^{\underline{a}_0} : ... : \lambda \underline{t}^{\underline{a}_m}])(\lambda \underline{t'}^{\underline{p}})^{k_\gamma} = 0.$$

For all w, $h_w([\lambda \underline{t}^{\underline{a}_0} : ... : \lambda \underline{t}^{\underline{a}_m}])(\lambda \underline{t'}^{\underline{p}})^{k_w}$ cannot cancel with any other term since no other term has $t_{n+1}^{k_w}$ in it, as all the k_w's are distinct.

Thus for all w, $h_w([\lambda \underline{t}^{\underline{a}_0} : ... : \lambda \underline{t}^{\underline{a}_m}])(\lambda \underline{t'}^{\underline{p}})^{k_w}$ must equal 0. This implies $h_w([\lambda \underline{t}^{\underline{a}_0} : ... : \lambda \underline{t}^{\underline{a}_m}]) = 0$ or $(\lambda \underline{t'}^{\underline{p}})^{k_w} = 0$. Since $(\lambda \underline{t'}^{\underline{p}})^{k_w} \neq 0$, we have that $h_w([\lambda \underline{t}^{\underline{a}_0} : ... : \lambda \underline{t}^{\underline{a}_m}]) = 0$. This implies that

$$\forall w,\ h_w(x_0, ..., x_m) \in I(X_P) = \langle g_1, ..., g_s \rangle \mathbb{C}[x_0, .., x_m].$$

Thus $\forall w, h_w = r_{w1}g_1 + ... + r_{ws}g_s$ where $r_{wc} \in \mathbb{C}[x_0, .., x_m]$, $\forall c$. Therefore

$$f = (r_{11}g_1 + ... + r_{1s}g_s)(x_{m+1}^{k_1}) + ... + (r_{\gamma 1}g_1 + ... + r_{\gamma s}g_s)(x_{m+1}^{k_\gamma})$$

Combining like terms and substituting $\beta_i = r_{1i}(x_{m+1}^{k_1}) + r_{2i}(x_{m+1}^{k_2}) + ... + r_{\gamma i}(x_{m+1}^{k_\gamma})$ we have that

$$f = \beta_1 g_1 + \beta_2 g_2 + ... + \beta_s g_s, \text{ where } \beta_i \in \mathbb{C}[x_0, ..., x_{m+1}],\ \forall i.$$

This implies $f \in \langle g_1, ..., g_s \rangle \mathbb{C}[x_0, ..., x_{m+1}]$.

$$\text{So } \langle g_1, ..., g_s \rangle \mathbb{C}[x_0, ..., x_{m+1}] \subseteq I(X_{pyr(P)}).$$

Thus $\langle g_1, ..., g_s \rangle \mathbb{C}[x_0, ..., x_{m+1}] = I(X_{pyr(P)})$. Therefore $I(X_P)$ and $I(X_{pyr(P)})$ have the same generators. □

Note that the standard n-simplex is a pyramid of the standard $(n-1)$-simplex and so this proof provides another way to show that $I(X_{\Delta_n}) = \langle 0 \rangle$.

Now recall from the last section that the product of two polytopes $P \subseteq \mathbb{R}^m$ and $Q \subseteq \mathbb{R}^n$ is $P \times Q = \{(p, q) \in \mathbb{R}^{m+n} \mid p \in P, q \in Q\}$ and that P×[0,1] is called the prism of P.

Proposition 3.19. *Let P be a full-dimensional lattice polytope in $\mathbb{R}^n$. There are two copies of $I(X_P)$ in $I(X_{P\times[0,1]})$.*

Proof. Let $P \cap \mathbb{Z}^n = \{\underline{a}_0, ..., \underline{a}_m\}$, so

$$\phi_P(\underline{t}) = [\underline{t}^{\underline{a}_0} : ... : \underline{t}^{\underline{a}_m}] \text{ where } \underline{t} = (t_1, ..., t_n) \in (\mathbb{C}^*)^n.$$

Thus $\operatorname{Im} \phi_P = \{[\lambda \underline{t}^{\underline{a}_0} : ... : \lambda \underline{t}^{\underline{a}_m}] \mid \lambda \neq 0,\ \underline{t} \in (\mathbb{C}^*)^n\}$.

$(P \times [0,1]) \cap \mathbb{Z}^{n+1} = \{(\underline{a}_0, 0), ..., (\underline{a}_m, 0), (\underline{a}_0, 1), ..., (\underline{a}_m, 1)\}$.
Let $\underline{t'} = (t_1, ..., t_n, t_{n+1}) \in (\mathbb{C}^*)^{n+1}$,

$$\begin{aligned} \phi_{P\times[0,1]}(\underline{t'}) &= [\underline{t'}^{(\underline{a}_0,0)} : ... : \underline{t'}^{(\underline{a}_m,0)} : \underline{t'}^{(\underline{a}_0,1)} : ... : \underline{t'}^{(\underline{a}_m,1)}] \\ &= [\underline{t}^{\underline{a}_0} : ... : \underline{t}^{\underline{a}_m} : \underline{t}^{\underline{a}_0} t_{n+1} : ... : \underline{t}^{\underline{a}_m} t_{n+1}]. \end{aligned}$$

And so we have,

$\operatorname{Im}\phi_{P\times[0,1]} = \{[\lambda \underline{t}^{\underline{a}_0} : ... : \lambda \underline{t}^{\underline{a}_m} : \lambda \underline{t}^{\underline{a}_0} t_{n+1} : ... : \lambda \underline{t}^{\underline{a}_m} t_{n+1}] \mid \lambda \neq 0, \underline{t} \in (\mathbb{C}^*)^n\}$.

Let $I(X_P) \subseteq \mathbb{C}[x_0, ..., x_m]$ and let $I(X_{P\times[0,1]}) \subseteq \mathbb{C}[x_0, ..., x_m, y_0, ..., y_m]$. Let $I(X_P)=\langle g_{x1}, ..., g_{xs}\rangle$. Since X_P is a projective toric variety its ideal is homogeneous and so we can assume that for all i, g_{xi} is homogeneous. We can express g_{xi} as $\sum_{j=0}^{N} \alpha_{ij}(\underline{x}^{\underline{\beta}_{ij}}) \in \mathbb{C}[x_0, ..., x_m]$ where $\alpha_{ij} \in \mathbb{C}$ and $\beta_{ij} \in \mathbb{N}^{m+1}$. Let $g_{yi} = \sum_{j=0}^{N} \alpha_{ij}(\underline{y}^{\underline{\beta}_{ij}}) \in \mathbb{C}[y_0, ..., y_m]$. So g_{yi} is also homogeneous. Let $\overline{g_{xi}} = \sum_{j=0}^{N} \alpha_{ij}(\underline{x}^{\underline{\beta}_{ij}}) \in \mathbb{C}[x_0, ..., x_m, y_0, ...y_m]$ and $\overline{g_{yi}} = \sum_{j=0}^{N} \alpha_{ij}(\underline{y}^{\underline{\beta}_{ij}}) \in \mathbb{C}[x_0, ..., x_m, y_0, ..., y_m]$.

We need to show (1) $\overline{g_{xi}} \in I(X_{P\times[0,1]})$ $\forall i \in \{1, ...s\}$ and (2) $\overline{g_{yi}} \in I(X_{P\times[0,1]})$ $\forall i \in \{1, ...s\}$.

(1) Note that $\forall i \in \{1, ..., s\}$,

$$\begin{aligned}\overline{g_{xi}}([\lambda \underline{t}^{\underline{a}_0} : ... : \lambda \underline{t}^{\underline{a}_m} : \lambda \underline{t}^{\underline{a}_0} t_{n+1} : ... : \lambda \underline{t}^{\underline{a}_m} t_{n+1}\,]) &= \overline{g_{xi}}([\lambda \underline{t}^{\underline{a}_0} : ... : \lambda \underline{t}^{\underline{a}_m} : 0 : ... : 0]) \\ &= g_{xi}([\lambda \underline{t}^{\underline{a}_0} : ... : \lambda \underline{t}^{\underline{a}_m}]) \\ &= 0,\end{aligned}$$

since $\overline{g_{xi}}$ has no y_0 through y_m terms and $g_{xi} \in I(X_P)$. Therefore by Corollary 3.11, $\overline{g_{xi}} \in I(X_{P\times[0,1]})$, $\forall i$. This implies that

$$\langle g_{x1}, ..., g_{xs}\rangle \mathbb{C}[x_0, ..., x_m, y_0, ..., y_m] \subseteq I(X_{P\times[0,1]}),$$

where $\langle g_{x1}, ..., g_{xs}\rangle \mathbb{C}[x_0, ..., x_m, y_0, .., y_m] = \{g_{x1}q_1 + g_{x2}q_2 + ... + g_{xs}q_s \mid q_k \in \mathbb{C}[x_0, ..., x_m, y_0, ..., y_m],\ 1 \leq k \leq s\}$.

(2) Let $d > 0$. Since $\overline{g_{yi}}$ has no x_0 through x_m terms and g_{yi} is homogeneous,

$$\begin{aligned}\overline{g_{yi}}([\lambda \underline{t}^{\underline{a}_0} : ... : \lambda \underline{t}^{\underline{a}_m} : \lambda \underline{t}^{\underline{a}_0} t_{n+1} : ... : \lambda \underline{t}^{\underline{a}_m} t_{n+1}\,]) &= \overline{g_{yi}}([0 : ... : 0 : \lambda \underline{t}^{\underline{a}_0} t_{n+1} : ... : \lambda \underline{t}^{\underline{a}_m} t_{n+1}\,]) \\ &= g_{yi}([\lambda \underline{t}^{\underline{a}_0} t_{n+1} : ... : \lambda \underline{t}^{\underline{a}_m} t_{n+1}\,]) \\ &= t_{n+1}^{d} g_{yi}([\lambda \underline{t}^{\underline{a}_0} : ... : \lambda \underline{t}^{\underline{a}_m}]) \\ &= 0.\end{aligned}$$

Therefore by Corollary 3.11, $\overline{g_{yi}} \in I(X_{P\times[0,1]})$, $\forall i$. This implies that

$$\langle g_{y1}, ..., g_{ys}\rangle \mathbb{C}[x_0, ..., x_m, y_0, ..., y_m] \subseteq I(X_{P\times[0,1]}),$$

Thus there are two copies of $I(X_P)$ in $I(X_{P\times[0,1]})$. □

Conjecture 3.20. If P and Q are lattice polytopes, $I(X_P)$ is contained in $I(X_{P\times Q})$ and $I(X_Q)$ is contained in $I(X_{P\times Q})$.

Remark 3.21. Notice that the conjecture is true for prisms since by Proposition 3.19, $I(X_P)$ is contained in $I(X_{P\times[0,1]})$ and $I(X_{[0,1]}) = \langle 0\rangle \subset I(X_{P\times[0,1]})$ by Propostion 3.14, since [0, 1] is a simplex.

Futher Reading:
If you would like to read more about the topics introduced in this paper, refer to [1], [5], and [6].

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References

1. Günter Ewald, *Combinatorial convexity and algebraic geometry*, Graduate Texts in Mathematics, vol. 168, Springer-Verlag, New York, 1996. MR **MR1418400 (97i:**52012)
2. Ewgenij Gawrilow and Michael Joswig, *polymake, a tool for the algorithmic treatment of convex polyhedra and finite simplicial complexes*, Available at http://www.math.tu-berlin.de/polymake/.
3. I. M. Gel′fand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1994. MR **MR1264417 (95e:**14045)
4. Daniel R. Grayson and Michael E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, Available at http://www.math.uiuc.edu/Macaulay2/.
5. Branko Grünbaum, *Convex polytopes*, second ed., Graduate Texts in Mathematics, vol. 221, Springer-Verlag, New York, 2003, Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler. MR **MR1976856 (2004b:**52001)
6. Günter M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer-Verlag, New York, 1995. MR **MR1311028 (96a:**52011)

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